Probabilistic Assortative Matching under Nash Bargaining*

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Abstract

This paper re-visits the canonical random search and matching model with Nash bargaining. By introducing *pair-specific production shocks*, our framework generates meeting-contingent match outcomes that are random. We provide a robust characterization of probabilistic matching patterns for any *non-stationary* environment, generalizing results by Shimer and Smith (2000). We find that, although their prediction of single-peaked preferences over meetings is robust, search frictions upset positive assortative matching across well-assorted pairs. As a second contribution, we show that the non-stationary random search matching model is a mean field game, and admits a representation as a system of forward backward stochastic differential equations. This representation affords a novel existence and uniqueness result, casting doubt on the robustness of multiple self-fulfilling equilibrium paths frequently reported in the literature.

Keywords: assortative matching, Nash bargaining, random search, non-stationary, mean field game
JEL classification: C73, C78, D81, E32

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1 Introduction

This paper provides a novel perspective on what is perhaps the most studied question in the theory of matching: when is matching assortative? Interest in this question is due to both the efficiency and distributive implications of assortative matching on the economy at large. For instance, when human and physical capital are complementary, a reallocation of more productive workers to capital-rich firms raises aggregate output. Regarding distributive concerns, a greater number of marriages between individuals of similar socio-economic status is known to exacerbate existing income inequality at the household level.

The theory of decentralized matching has been spurred by Becker (1973), who formulates the first key insight within a frictionless competitive market: assortative matching arises due to complementarities in output between similar individuals. However, frictionless search oftentimes presents an unattainable idealization of matching markets; unemployment spells are the most obvious case in point that search frictions can be sizeable. In this paper, we study sorting in matching markets where random search frictions impede the instantaneous clearing of supply and demand. Thereby, a given individual is willing to match with a range of individuals as opposed to a single ideal partner. As in the Diamond-Mortensen-Pissarides paradigm of the labor market, we replace the Walrasian auctioneer by Nash bargaining.[1]

Our understanding of sorting in this framework is due to Shimer and Smith (2000), and complementary work by Atakan (2006).[2] Their main contribution is to identify sufficient conditions on output for which (a set based notion of) positive assortative matching arises despite the presence of search frictions: higher types match with intervals of higher types.[3] In contrast, we argue that positive assortative matching is not a robust feature of matching markets —even when match output exhibits a high degree of complementarity. In this article we identify economic forces that prevent positive assortative matching from occurring when search frictions are sizeable. To make this point, we propose a framework in which match outcomes are probabilistic: no match is inconceivable, but those pairs who produce a greater match surplus are more likely to match. By comparison, in the deterministic framework of Shimer and Smith (2000) and Atakan (2006) meeting-contingent match probabilities are binary, $0-1$.

The difference between deterministic and probabilistic matching patterns can be best appreciated visually. Figure 1 (left) illustrates the set of matching pairs in a deterministic environment where Shimer and Smith (2000)'s set based notion of assortative matching obtains: matching sets are increasing in type. Conversely, figure 1 (center) illustrates meeting-contingent match probabilities in a probabilistic environment, where warmer colors correspond to higher probabilities of matching. The shaded region in figure 1 (right)

[1]The Nash bargaining solution in search and matching models is also referred to as the transferable utility (TU) paradigm in the literature; in contrast, a model where match payoffs are exogenously given (analyzed in Bonneton and Sandmann (2019)) is referred to as the non-transferable utility (NTU) paradigm.

[2]See Chade et al. (2017) for an overview of the literature.

[3]Common to both is the well-known condition from Becker (1973): supermodularity. In addition, Shimer and Smith (2000) require more stringent complementarity conditions on the curvature of $f(x, y)$, as well as boundary conditions that translate into weak self-preference among the lowest and the highest types; $f(x, y) = xy$ satisfies all of their conditions (as first studied by Lu and McAfee (1996)).

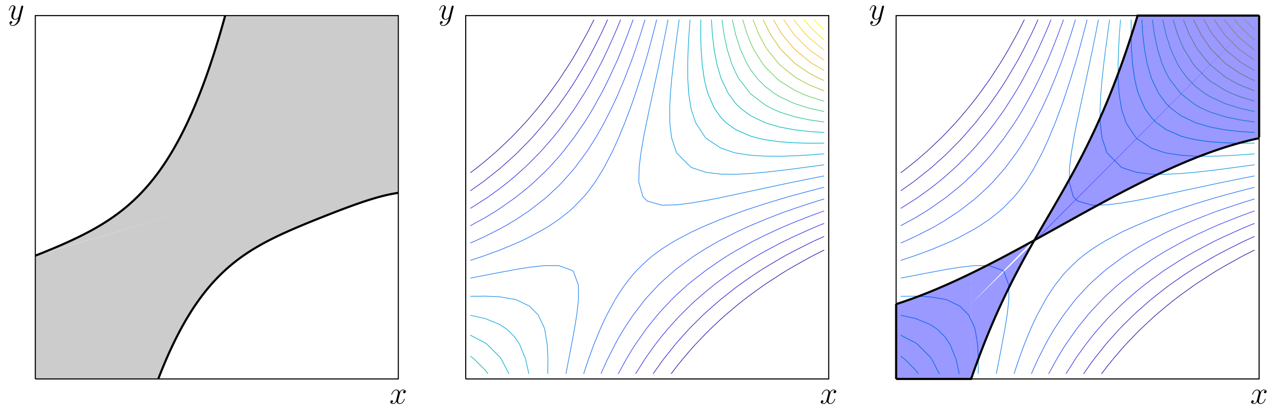

Figure 1: Deterministic and probabilistic matching patterns.

depicts the pairs of types for whom the (herein introduced) general notion of *probabilistic assortative matching* (PAM) fails: PAM requires that probabilistic level lines, depicting pairs of types that match with equal probability, are non-decreasing in either type. This is visibly not the case here. We call this a failure of assortative matching, for it implies that for all pairs of types $x_1 < x_2$ and $y_1 < y_2$ in this region, one of the assortative pairs (x_1, y_1) or (x_2, y_2) has a lower probability of matching than both non-assortative pairs (x_1, y_2) and (x_2, y_1). This failure can even arise even when (x_1, y_1) and (x_2, y_2) is the assortative assignment absent search frictions, so that pairs of perfect complements need not be the ones matching with the greatest frequency.

The failure of PAM in the presence of search frictions is not an artefact of specific modeling choices other than Nash bargaining; it is common to all random-search matching models in which meeting-contingent match probabilities are increasing in the match surplus. Under Nash bargaining, surplus rises in the output generated, yet falls in both agents' option values of search. Only the latter vary with the amount of search frictions present in the economy. Search frictions impede assortative matching from occurring because frictions disproportionately erode the value of search and hence the bargaining power of more productive agents. This is because more productive agents have greater opportunity costs of time due to discounting. Additionally, the initial increase in the opportunity cost of time further feeds back into payoffs: being more adversely affected by search frictions translates into a comparatively lower bargaining power vis-a-vis other agents. As a result, unproductive agents prioritize matching with productive agents. At the extreme when the market is thin, agents unanimously exhibit the greatest match surplus when matched with the most productive individuals. In our model, this translates one-to-one into greater meeting-contingent match probabilities with the most highly ranked individuals. In line with our visual representation, this implies that the assortative pair (x_1, y_1) exhibits a lower meeting-contingent match probability than the less assortative pairs (x_1, y_2) and (x_2, y_1).

If not assortatively, how do agents sort into pairs? Our first main result (theorem 3) provides a general characterization of matching patterns under known complementarity conditions —regardless of the level of search frictions present in the economy. We show that sorting takes the qualitative features depicted in figure 1 (center and right). First, meeting-contingent match probabilities are single-peaked: each agent type has an ideal partner's type, and meeting-contingent match probabilities are greater for those agents more closely resembling the ideal partner's type. Furthermore, more highly-ranked types have more highly-ranked ideal partners, an implication of single-crossing. In figure 1 (right) there are two curves enclosing the pairs of types for whom PAM fails; the more gradual curve depicts ideal partners' types of agents types x, whereas the steeper curve depicts ideal partners' types of agents types y. In the region enclosed by both curves, hereafter the enclosure, level lines are decreasing so that PAM fails, whereas in the outer region level lines are increasing so that PAM obtains.[4] The pairs of types enclosed by both curves are well-assorted, in the sense that either they comprise only low or high types, or their meeting-contingent match probability is large.

Our modeling choices emphasize two elements of the theory, that have been abstracted from in much of the preceding empirical and theoretical literature alike, but that we view as essential to correctly identify sorting. Our objective is to provide a theory simple enough to allow for theoretical insights into sorting, yet rich enough to be plausibly at the origin of empirically observed matching patterns.

First, we view match output as random, not deterministic. Specifically, as in Choo and Siow (2006), we take output to be the sum of a type-specific deterministic component $f(x, y)$ and a pair-specific production shock ξ.[5] When two agents meet, they match if the realization of their production shock ξ is large enough, i.e., if $f(x, y)+\xi$ exceeds their joint option value of search. Thereby, match outcomes upon meeting are *random*. In particular, pairs of types that produce a larger surplus match with strictly greater probabilities.

Secondly, the economy is non-stationary. This means that the size and composition of the search pool fluctuates endogenously over time; equilibrium match acceptance decisions and individual payoffs adjust accordingly. Non-stationarity is arguably a first-order concern to sorting. As has been well-documented, wages vary over the business cycle and are higher during periods of increased match creation, i.e., booms.[6] Here we report results that make no distinction between non-stationary and stationary environments thus far studied in the literature.[7]

Studying sorting at this level of generality is non-trivial. As pointed out by Smith (2011), "even the simplest non-stationary models can be notoriously intractable." We circumvent the tractability issues that come with non-stationary dynamics by constructing tight bounds on the difference in the value of search between two agents. In lemma 1,

[4] In the paper we think of (and establish an equivalence of) meeting-contingent match probabilities in terms of intuitive and empirically relevant ordinal properties of individual preferences over meetings: agent type x prefers to meet y_2 over y_1 if it gives x a greater expected match payoff. In our model x prefers to meet y_2 over y_1 if and only if (x, y_2) have a greater meeting-contingent match probability than (x, y_1). Thus refer to ideal partner's types also as preferred partner's types.

[5] In the literature review we provide a detailed empirical justification as to why we need production shocks to understand matching patterns.

[6] Lise and Robin (2017) propose and estimate a non-stationary random search matching model of workers and firms with on-the-job search when dynamics are driven by exogenous shocks. A key finding is that the estimated matching sets during lower productivity states are smaller. This leads to more assortative matching in hiring (from unemployment) during recessions.

[7] We do not suggest that non-stationarity plays a limited role under bargaining. Differences in sorting between stationary and non-stationary environments remain to be explored. It may well be the case that complementarity conditions are needlessly strong in the steady state as is the case in the NTU paradigm: as shown by Bonneton and Sandmann (2019), the well-known (steady state) results on assortative matching from Smith (2006) and Morgan (1994) only obtain under stronger complementarity conditions when the economy is non-stationary. This derives from the risk of worsening match prospects, inherent to non-stationary dynamics.

following Bonneton and Sandmann (2019), we apply a revealed preference argument. The underlying idea is to let one agent replicate someone else's matching decisions; such "mimicking" strategy must be weakly dominated due to the intratemporal efficiency of Nash bargaining. As a second step, lemma 3 uses a novel inductive reasoning over the revealed preference argument, giving rise to perhaps the most beautiful proof in this paper, *the matryochka dolls*.

Our second main contribution is to prove the existence of a unique non-stationary equilibrium (theorem 1) in a framework where economic fundamentals are defined in utmost generality.

What is a non-stationary equilibrium? The notion of equilibrium always entails an optimization problem on the side of the agents. Under Nash bargaining, this is easily resolved: two agents match whenever ex-post output $f(x, y) + \xi$ exceeds the joint option value of search. However, this solution requires knowledge of agents' option values of search. The difficulty in establishing an equilibrium arises from solving an intricate feedback loop between the value of search and population dynamics. Agents' expectations and decisions must conform with the population dynamics to which they give rise. As the population evolves, so do agents' expectations. Accordingly, we define an equilibrium as a coupled system characterizing jointly forward-looking population dynamics and backward-looking agents' option values of search under the Nash bargaining solution. Lasry and Lions (2007) call such coupled system and the mathematical techniques that pertain to it a *mean field game*.[8]

Here we find it suitable to consider a stochastic economy: the evolution of the search pool, while dependant on agents' matching decisions, is in addition subject to random entry. Random entry allows us to draw on techniques from stochastic calculus, notably the martingale representation theorem and Ito's lemma. The key result (corollary 1) is to show that our equilibrium can be represented as a system of *forward backward stochastic differential equations (FBSDEs)*. The theory of FBSDEs is a powerful tool in environments where laws governing the population dynamics and the value of search, as described by the Kolmogorov and Hamilton-Jacobi-Bellman partial differential equations, are not as smooth as one would like them to be.[9] Our representation of the stochastic environment as a system of FBSDEs allows us to (subject to identifying the relevant economic regularity conditions such as Lipschitz continuity and boundedness of parameters) resolve existence and uniqueness with a well-posedness theorem due to Delarue (2002).

The remainder of this paper is organized as follows. Section 2 lays out the model. Section 3 establishes existence and uniqueness. Section 4 studies properties of agents' preference. Section 5 characterizes equilibrium sorting. Section 6 concludes. Proofs, if not found in the text, as well as the related literature are relegated to the appendix.

[8] The name comes from the"Mean Field theory" in physics; it is an analogy to the continuum limit taken in which one approximates large systems of interacting particles by assuming that these interact only with the statistical mean of other particles.

[9] In the deterministic environment studied in Bonneton and Sandmann (2019) we went to great lengths to circumvent discontinuities inherent to random search matching.

2 The model

We develop a continuous-time, finite-horizon $[0, T]$ matching model in which continua of ex-ante heterogeneous agents engage in time-consuming and haphazard search for one another. Any two agents that meet may decide whether to form a match, whereupon they exit the search pool and divide match output according to Nash bargaining. Otherwise they continue waiting for a more suitable partner.

2.1 Set-up

There are two distinct populations, employment-seeking workers and firms with vacancies say, each comprising a continuum of ex-ante heterogeneous agents. Agents of type $x \in X$ seek to match with agents of type $y \in Y$. We take X and Y to be finite, ordered and disjoint sets, i.e., $X = \{1, ..., N^X\}$ and $Y = \{N^X + 1, ..., N\}$, where $N = N^X + N^Y$. In what follows we usually take the viewpoint of an agent type $x \in X$. It goes without saying that symmetric constructions apply for agent types $y \in Y$.

The mass of agent types x in the search pool at time t is given by a positive number $\mu_t(x) \in \mathbb{R}_+$. The environment we study is non-stationary in the sense that the mass and composition of the search pool $\mu_t = ((\mu_t(x))_{x \in X}, (\mu_t(y))_{y \in Y}) \in \mathbb{R}_+^N$ changes over time. The search pool population μ_t and agents' continuation values of search, V_t, introduced at a later stage, are the economy's Markovian state at time t.

Output and matching decisions

Two agents that match with one another produce a lump sum match output. Match output is the sum of deterministic $f(x, y)$ and random ξ:

$$f(x, y) + \xi \quad \text{where} \quad \xi \sim \Xi_t.$$

We refer to $f(x, y)$ as the *ex-ante match output*, because it is manifest before a meeting has taken place. Symmetrically, we refer to $f(x, y) + \xi$ as *ex-post match output*, because it is only manifest once a meeting has taken place. ξ is distributed independently and identically across all agent pairs according to a state- and time-variant cumulative distribution function Ξ_t. The subscript t subsumes throughout the dependence on both time and the economy's Markovian state variable.

As will become apparent soon, individual match payoffs are increasing in the realization of ξ. It follows that in equilibrium joint match acceptance decisions conditional on meeting are given by a time- and state-varying threshold rule $\theta_t(x, y)$. The probability that ξ exceeds $\theta_t(x, y)$ determines the probability $m_t^\theta(x, y)$ that two agent types x and y that meet at time t match with one another:

$$m_t^\theta(x, y) = \int_{\theta_t(x,y)}^{+\infty} \Xi_t(d\xi). \tag{1}$$

We call this the *probabilistic matching function*. Observe that absent production shocks

ξ, match probabilities are binary, either zero or one.

Meetings

Meetings follow a time- and state-varying Poisson point process where $\lambda_t(y)$ is the rate at which any agent type x meets an agent type y at time t. As is common in the literature, the meeting technology is anonymous. This means that one's own type x has no bearing on the probability of meeting other agents. Concordantly, we do not impose technological assumptions that favor assortative matching.

Remark 1. *Any anonymous meeting rate can be represented as*

$$\lambda_t(y) = \beta_t \, \mu_t(y)$$

for some time-and state-dependent β_t, common to all types.

For instance, search is quadratic when $\beta_t = 1$, search is linear when $\beta_t = 1/\sum_z \mu_t(z)$.[10] The representation derives from internal consistency of the model, which requires that the flow number of meetings between two agent types $x \in X$ and $y \in Y$ can be symmetrically computed taking either agent type x's or y's perspective: $\lambda_t(y)\mu_t(x) = \lambda_t(x)\mu_t(y)$.

Entry, exit and population dynamics

Population dynamics are governed by entry and exit:

- The rate at which agent types x exit the search pool (for a given threshold rule θ_t) is given by x's individual hazard rate times the total mass of types x in the search pool, $\mu_t(x)$:

$$\sum_{y \in Y} \lambda_t(y) m_t^{\theta}(x, y) \mu_t(x).$$

- The rate at which agent types x enter into the search pool is described by a geometric Brownian motion with endogenous parameters,

$$\mu_t(x)\eta_t(x)dt + \mu_t(x)\sigma_t(x)dBt(x).$$

  Here $\eta_t(x)$ denotes the time-and state-varying drift and $\sigma_t(x)$ the time-and state-varying volatility; $B = ((B(x)_t), (B(y)_t))_{t \geq 0}$ is an N-dimensional $\mathcal{F}_t$-adapted standard Brownian motion in the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$ satisfying the usual conditions.

Entry and exit aggregate into a forward stochastic differential equation where the initial population in the search pool is $\mu_0 \in \mathbb{R}_+^N$:

$$\mu_t(x) = \mu_0(x) + \int_0^t \big[- \sum_y \lambda_\tau(y) \mu_\tau(x) m_\tau^{\theta}(x, y) + \mu_\tau(x)\eta_\tau(x) \big] d\tau + \int_0^t \mu_\tau(x)\sigma_\tau(x) dB_\tau(x). \tag{2}$$

[10] Our existence result requires that $\lambda_t(y)$ be bounded, i.e., $\beta_t \leq K^\beta / \sum_z \mu_t(z)$ for some $K^\beta > 0$. This formally rules out quadratic search, but only so for type distributions which realize with arbitrarily small probabilities.

Nash bargaining and the value of search

If two agent types x and y decide to match, they share ex-post output according to Nash bargaining: agents receive as payoff their respective continuation value of search $V_t(x)$ or $V_t(y)$ plus a share of the remaining output $S_t(x, y) + \xi$. Here

$$S_t(x, y) \equiv f(x, y) - V_t(x) - V_t(y)$$

denotes ex-ante surplus. Under this interpretation, standard in the search and matching literature, the option value of search is used as the outside option in the bargaining game. Agent type x's expected match payoff for given threshold rule θ_t is then given by

$$\pi_t^\theta(x, y) = V_t(x) + \frac{\alpha_t^X}{m_t^\theta(x, y)} \int_{\theta_t(x,y)}^{+\infty} \left[S_t(x, y) + \xi\right] \Xi_t(d\xi), \tag{3}$$

where α_t^X denotes the share of surplus accruing to types $x \in X$ with $\alpha_t^X + \alpha_t^Y \leq 1$[11] (and $m_t^\theta(x, y) > 0$ by assumption). It is easy to see that said payoff is maximal if matches are consummated whenever ex-post match surplus is non-negative, i.e.

$$\theta_t^*(x, y) = -S_t(x, y).$$

In equilibrium, agents are forward-looking and weigh the expected discounted match payoff against the immediate match payoff at hand. The remainder of this section asserts that $\theta_t^*(x, y) = -S_t(x, y)$ also maximizes time t individual expected discounted match payoffs (where the meeting rate λ_τ and value of search V_τ for $\tau > t$ is exogenously given), and defines equilibrium matching thresholds.

To formalize this claim, fix $(\lambda_\tau)_{\tau>t}$ and $(V_\tau)_{\tau>t}$ (objects which are beyond the control of an individual agent), and define $r_t^\theta(y|x)$ the earliest time $r \geq t$ such that type x meets a type y and draws $\xi \geq \theta_r(x, y)$:

$$r_t^\theta(y|x) = \min\left\{r \geq t : \text{ a given } x \text{ meets some } y \text{ s.t. } \xi \geq \theta_r(x, y)\right\}.$$

With threshold rule θ in place, the probability that x matches with some y during time interval $[t, \tau]$ is denoted by

$$P_t^\theta(\tau)(y|x) = \text{Prob}\left[r_t^\theta(y|x) \leq \tau \text{ and } r_t^\theta(y|x) \leq r_t^\theta(y'|x) \,\forall\, y'\right],$$

and agent type x's expected discounted match payoff is

$$W_t^\theta(x) \equiv \mathbb{E}\Big[\int_t^T e^{-\rho(\tau-t)} \sum_{y\in Y} \pi_\tau^\theta(x, y) P_t^\theta(d\tau)(y|x) + e^{-\rho(T-t)}\big(1 - \sum_{y\in Y} P_t^\theta(T)(y|x)\big) h_T \big| \mathcal{F}_t\Big].$$

(h_T is the μ_T-dependent terminal payoff, common to all agents, which accrues if unmatched at time T. In the simplest interpretation, set this to zero.)

If the optimal individual stopping rule is to accept a match whenever $S_t(x, y) + \xi \geq 0$,

[11] This encompasses typically considered bargaining weights $\alpha_t^X = \alpha_t^Y = \frac{1}{2}$.

optimal stopping exhibits a striking symmetry: agent type x will only want to match with agent type y, if agent type y wants to match with agent type x; two agents would never disagree. If so, the individually optimal matching threshold and the equilibrium matching rule coincide. That this is indeed the case is established by the following lemma.

Lemma 1 (Intratemporal efficiency)**.** $W_t^{\theta^*}(x) = \sup_\theta W_t^\theta(x)$ *where* $\theta_t^*(x,y) = -S_t(x,y)$.

The result derives from the efficiency of the Nash bargaining sharing rule: if there exist gains from matching, the transfer rule is such that both parties stand to benefit. We henceforth omit all superscripts as equilibrium matching thresholds will throughout be governed by θ_t^*. The time t value of search is given by

$$V_t(x) \equiv \sup_\theta W_t^\theta(x). \tag{4}$$

As we go forward, a final remark is in order.

Remark 2. *The defining quantities of the model, notably* μ_t *and* V_t*, are stochastic processes and depend on the particular realization of the sample path of* $(B_\tau)_{0\leq\tau\leq t}$*. Stipulated inequalities must hold for all realizations* $\omega \in \Omega$*, unless we suppose by contradiction; by this we mean that there exists* $\omega \in \Omega$ *such that the reverse inequality holds at time* t *for the event* $\cap_{F\in\mathcal{F}_t:\omega\in F}F$*.*

3 Equilibrium existence and uniqueness

3.1 Definition of equilibrium

An equilibrium is a characterization of matching patterns, i.e., specifies who matches with whom. Matching patterns derive from individual choices. Lemma 1 asserts that optimal choices adhere to the non-negative ex-post surplus rule: two agents match if and only if ex-post surplus is non-negative. It follows that matching decisions are a function of agents' value of search, which in turn depends on the evolution of the search pool.

Given an optimal matching threshold θ^*, an equilibrium is taken to be double (μ, V) where μ is given by (2) and V is given by (4).

Noticeably, the evolution of the search pool and the agents' value of search interact with one another: agents' evaluate their value of search under expectations which must conform to the population dynamics. Population dynamics exhibit a higher hazard rate and increased match creation if the value of search is low. This creates a feedback loop between population dynamics which move forward from the origin of time and values of search which are constructed backward from a final time T. The forward-backward structure is a feature of virtually all dynamic general equilibrium models under rational expectations. Adopting terminology from physics, some authors refer to such a system as a mean field game.

In what follows we seek to represent (μ, V) as a system of forward-backward stochastic differential equations (FBSDEs). This allows to connect our problem of equilibrium existence and uniqueness to an established literature in mathematics which studies the well-posedness of systems of FBSDEs.

3.2 System of FBSDEs

The population dynamics were given by a forward stochastic differential equation, looking forward from some initial condition μ_0. For analytical purposes we find it more convenient to work with a monotone transformation of $\mu_t(x)$ instead. Thus define $\gamma_t(x) = \ln \mu_t(x)$. Ito's lemma establishes that

$$\gamma_t(x) = \gamma_0(x) + \int_0^t \Big[-\sum_{y \in Y} \beta_\tau e^{\gamma_\tau(y)} m_\tau(x,y) + \eta_\tau(x) - \frac{\big(\sigma_\tau(x)\big)^2}{2} \Big] d\tau + \int_0^t \sigma_\tau(x) dB_\tau,$$

where $\gamma_0(x) = \ln \mu_0(x)$. The motivation for this transformation is straightforward: $d\gamma_t(x)$ can be thought of as the flow percentage change of agents in the search pool whereas $d\mu_t(x)$ captures the absolute flow change in the mass of agent types x. In that we focus on the former (joint with the assumption that $\sigma_t(x)$ is element in some positive bounded interval), we rule out that (i) the mass of agents searching becomes negative due to large, negative realizations of random entry, and (ii) accommodate a search pool which may grow unboundedly large as long as the expected growth rate remains bounded.

We now establish that the continuation value of search can be expressed as a backward stochastic differential equation. This is called a backward stochastic differential equation, because the solution is constructed backwards from the terminal time T.

Lemma 2. *The value of search is, given μ_t, the solution to the following backward stochastic differential equation:*

$$V_t(x) = h_T + \int_t^T \Big[\sum_{y \in Y} \big[\pi_\tau(x,y) - V_\tau(x)\big] \beta_\tau e^{\gamma_\tau(y)} m_\tau(x,y) - \rho V_\tau(x) \Big] d\tau - \int_t^T Z_\tau(x) \cdot dB_\tau.$$

The lemma follows from an application of the martingale representation theorem which can be likened to the fundamental theorem of calculus in stochastic analysis. A transformation of $V_t(x)$ can be shown to be a martingale. Then $Z_t(x)$ can be thought of as its derivative.

To summarize, this establishes that an equilibrium is a solution to the following system of forward-backward stochastic differential equations.

Corollary 1. $(\mu, V) \equiv (\exp \gamma, V)$ *is an equilibrium if and only if* (γ, V, Z) *solves the following system of forward-backward stochastic differential equations*

$$\begin{cases} \gamma_t(x) = \gamma_0(x) + \int_0^t \Big[-\sum_{y \in Y} \beta_\tau e^{\gamma_\tau(y)} m_\tau(x,y) + \eta_\tau(x) - \frac{(\sigma_\tau(x))^2}{2} \Big] d\tau + \int_0^t \sigma_\tau(x) dB_\tau \\ V_t(x) = h_T + \int_t^T \Big[\sum_{y \in Y} \big[\pi_\tau(x,y) - V_\tau(x)\big] \beta_\tau e^{\gamma_\tau(y)} m_\tau(x,y) - \rho V_\tau(x) \Big] d\tau - \int_t^T Z_\tau(x) \cdot dB_\tau. \end{cases} \tag{E}$$

where $\gamma_0 = \exp \mu_0$ *and* h_T *are given.*

3.3 Regularity conditions

In most applications economic fundamentals $\beta, \eta, \Xi, \alpha, \sigma, h$ will be time-and state-invariant parameters. A theory which treats those as parameters is meaningful, no intuition is lost focusing on this case, and numerical examples will consider them as such. For example, consider

$$\big(\eta_t, \alpha_t, \sigma_t, h_T\big) \equiv \big(0, \frac{1}{2}, \epsilon, 0\big).$$

Well-posedness and therefore the existence of a unique equilibrium will obtain with these parameter values as long as the meeting rate is bounded (as is readily implied by linear search or truncated quadratic search for instance), and Ξ_t captures a standard bounded distribution (such as a truncated normal with support exceeding deterministic output).

However, as theorists we strive for generality. As our theoretical results on sorting and equilibrium existence and uniqueness do not require economic fundamentals to be time- and state-invariant, we posit a more general relationship between economic fundamentals and both time t and state variables. Because it affords a cleaner representation, we consider here the log of the population, i.e., γ_t not μ_t. The relevant time t Markovian states are given by $\gamma_t \in \mathbb{R}^N$ and $V_t \in \mathbb{R}^N$. We will subsequently and to the extent that it is necessary impose regularity conditions that govern this relationship.

$$\begin{aligned}
\beta : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+ &\quad \text{where} \quad \beta_t \equiv \beta(t, \gamma_t, V_t) \\
\eta : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+^N &\quad \text{where} \quad \eta_t \equiv \eta(t, \gamma_t, V_t) \\
\alpha^X, \alpha^Y : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to [0,1] &\quad \text{where} \quad \alpha_t^X \equiv \alpha^X(t, \gamma_t, V_t),\ \alpha_t^Y \equiv \alpha^Y(t, \gamma_t, V_t) \\
\Xi : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \{cdf\text{s on } \mathbb{R}\} &\quad \text{where} \quad \Xi_t \equiv \Xi(t, \gamma_t, V_t) \\
\sigma : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+^N &\quad \text{where} \quad \sigma_t \equiv \sigma(t, \gamma_t, V_t) \\
h : \mathbb{R}^N \to \{\mathcal{F}_T\text{-measurable } rv \text{ on } \mathbb{R}\} &\quad \text{where} \quad h_T \equiv h(\gamma_T)
\end{aligned}$$

Here we write $\beta_t \equiv \beta(t, \gamma_t, V_t)$ to make explicit that whenever we write β_t we subsume the dependence of the meeting rate on the mass of agents searching at time t as well as their time t option value of search. The same remark applies for the functional relationships governing all other fundamentals.

We make the following assumption with regard to $\beta, \eta, \alpha, \Xi, \sigma, h$. For brevity of notation, denote $\Theta_t = (t, \gamma_t, V_t)$.

Assumption 1 (regularity condition).

- *β and η are non-negative, continuous in γ_t and Lipschitz continuous with constants L^λ and L^η respectively in V_t;*
- *there exists a constant $K^\beta > 0$ such that $\beta_t(\Theta_t) \leq \frac{K^\beta}{\sum_{z \in X \cup Y} \exp \gamma_t(z)}$;*
- *there exists a constant $K^\eta > 0$ which bounds η from above;*
- *Both α^X and α^Y are in $[0,1]$ with $\alpha^X + \alpha^Y \leq 1$, Lipschitz continuous in γ_t with constant L^α and continuous in V_t;*
- *Ξ_t admits a density Ξ'_t with bounded support $[\underline{\xi}, \overline{\xi}]$;*
- *Ξ is Lipschitz continuous in both (γ_t, V_t) and ξ with constant L^ξ, i.e. $(\gamma_t, V_t) \mapsto \Xi(t, \gamma_t, V_t)(\xi)$ is Lipschitz continuous for all $t \in [0,T]$ and $\xi \in [\underline{\xi}, \overline{\xi}]$, and $\xi \mapsto \Xi(t, \gamma_t, V_t)(\xi)$ is Lipschitz continuous for all $t \in [0,T]$ and $(\gamma_t, V_t) \in \mathbb{R}^N \times \mathbb{R}^N$;*
- *Ξ' is Lipschitz continuous in (γ_t, V_t) with constant L^ξ, i.e. $(\gamma_t, V_t) \mapsto \Xi'(t, \gamma_t, V_t)(\xi)$ is Lipschitz continuous for all $t \in [0,T]$*

- *σ is continuous in (t, γ_t, V_t) and bounded from below and above by strictly positive constants $\underline{\sigma}$ and $\overline{\sigma}$;*
- *There exists a non-negative constant L^σ such that $||\sigma(t, \gamma_t, V_t) - \sigma(t, \gamma_t', V_t')|| \leq L^\sigma(||\gamma_t - \gamma_t'||^2 + ||V_t - V_t'||^2)$ uniformly for all $(t, \gamma_t, V_t) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N$;*
- *h is bounded and Lipschitz continuous with constant L^h.*

For analytical convenience, we also require that the support of Ξ_t admits large positive and negative values with sufficiently flat tails, such that $m_t(x, y)$ is always bounded away from zero and one.

3.4 Existence and uniqueness of equilibrium

With assumption 1 in place, we are ready to establish the existence of a unique equilibrium.

Theorem 1. *If regularity condition 1 is in place, there exists a unique equilibrium.*

Our existence result derives from an application of a deep theorem on the well-posedness of systems of FBSDEs due to Delarue (2002). Refer to the appendix and related literature for further discussion.

4 Single-peaked preferences and single-crossing

Our definition of probabilistic positive assortative matching, as defined in the next section, can be equivalently recast in terms of ordinal properties of preferences. We begin our analysis by studying said preferences. We write $y_2 \succsim_t^x y_1$ when agent type x prefers to meet y_2 over y_1 at time t. A utility representation of said preference relation is readily given by the expected payoff conditional on meetings: $y \mapsto V_t(x) + m_t(x, y)\big(\pi_t(x, y) - V_t(x)\big)$. When one is certain to meet a type y at time t, one always retains one's option value of search $V_t(x)$. Moreover, with probability $m_t(x, y)$ the productivity shock ξ will be large enough for agent types x and y to form a match, in which case agent type x receives the expected match payoff $\pi_t(x, y)$.

We posit sufficient conditions on the degree of complementary of the ex-ante output, f, under which the preference relation $\succsim_t^x$ satisfies two, arguably classical, ordinal properties: (i) *single-crossing* and (ii) *single-peaked preferences.* Preferences exhibit single-crossing if for two possible types $y < y'$, there exists a threshold type $\hat{x}$ such that all types x smaller than $\hat{x}$ prefer to meet y, and all other types x prefer to meet y'. This conveys the idea that higher types x prefer to meet higher types y. Preferences are single-peaked, if each agent type x has an ideal partner, denoted $y_t(x)$, such that whenever agent type x compares two partners that are both to the right or to the left of $y_t(x)$, she strictly prefers whichever partner is closest to $y_t(x)$.

Since both (i) and (ii) are ordinal properties of the preference relation $\succsim_t^x$, they hold true for any utility representation, three of which are of interest:

Remark 3. *The preference relation $\succsim_t^x$ is represented by any of the following:*

- *the expected payoff conditional on meetings, $V_t(x) + m_t(x, y)\big(\pi_t(x, y) - V_t(x)\big)$,*
- *the probabilistic matching function, $m_t(x, y)$,*

- *the ex-ante surplus, $S_t(x,y)$,*

for one is a monotonic transformation of the other.[12]

Note that this remark relies on the assumption that idiosyncratic shocks are independently and identically distributed across agents. In what follows, we prove single-crossing and single-peaked preferences for the ex-ante surplus, $y \mapsto S_t(x,y)$. This ensures that the expected payoff conditional on meetings as well as the probabilistic matching function also satisfy (i) and (ii). Refer to figure 2 for an illustration of these properties for two realizations of random ex-ante surplus $S_t(x_1,\cdot)$ and $S_t(x_2,\cdot)$.

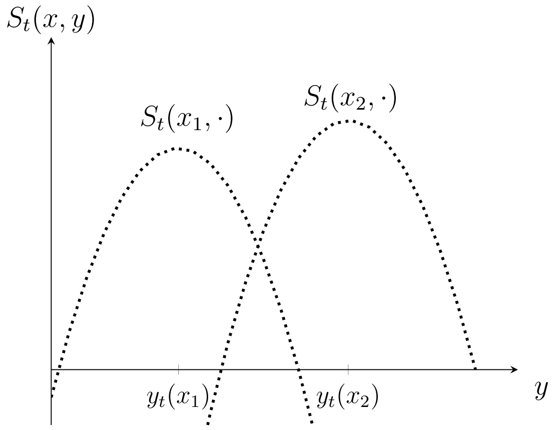

Figure 2: Single-peaked preferences and single-crossing

4.1 Single-crossing

We begin our analysis with single-crossing. This property stipulates that higher types x exhibit a preference for meeting with higher types y, which is suggestive of preferences for positive assortative matching.

Definition 1. *$S_t(x,y)$ satisfies single-crossing if for any two $x_1 < x_2$ and $y_1 < y_2$, both*

$$S_t(x_2,y_1) - S_t(x_1,y_1) \geq 0 \quad \Rightarrow \quad S_t(x_2,y_2) - S_t(x_1,y_2) > 0,$$
$$S_t(x_1,y_2) - S_t(x_1,y_1) \geq 0 \quad \Rightarrow \quad S_t(x_2,y_2) - S_t(x_2,y_1) > 0.$$

In a similar vein, $S_t(x,y)$ satisfies weak single-crossing if

$$S_t(x_2,y_1) - S_t(x_1,y_1) > 0 \quad \Rightarrow \quad S_t(x_2,y_2) - S_t(x_1,y_2) \geq 0,$$
$$S_t(x_1,y_2) - S_t(x_1,y_1) > 0 \quad \Rightarrow \quad S_t(x_2,y_2) - S_t(x_2,y_1) \geq 0.$$

In particular, it obtains when the ex-ante match output f is supermodular.

[12]This is because we can write the expected payoff conditional on meetings as an increasing function of the ex-ante surplus, $S_t(x,y) \mapsto V_t(x) + \int_{-S_t(x,y)}^{+\infty} \alpha_t^X (S_t(x,y) + \xi_t)\, \Xi_t(d\xi)$; likewise for the probabilistic matching function, $S_t(x,y) \mapsto 1 - \Xi_t(-S_t(x,y))$.

Definition 2 (supermodular). *Ex-ante match output f is supermodular if for any two $x_2 > x_1$ and $y_2 > y_1$ we have*

$$f(x_2, y_2) + f(x_1, y_1) > f(x_1, y_2) + f(x_2, y_1).$$

Supermodularity is a well-known property in the theory of matching. In a frictionless matching market the type with whom one matches is also the type with whom one generates the greatest match payoff. Under supermodularity, Becker (1973) shows that said type is non-decreasing in one's own type. The next proposition generalizes this result to an environment with search frictions.[13]

Proposition 1 (Single-crossing). *Supermodularity and single-crossing relate as follows:*

1. *$f(x, y)$ is supermodular,*
1. $\Leftrightarrow$ *2. $S_t(x, y)$ is supermodular,*
2. $\Rightarrow$ *3. $S_t(x, y)$ satisfies single-crossing,*
3. $\Rightarrow$ *4. there exists $\hat{y}_t(x) \in \arg\max_{y \in Y} S_t(x, y)$ which is non-decreasing in x.*

Proof. Recall that $S_t(x, y) = f(x, y) - V_t(x) - V_t(y)$ from which *1.* $\Leftrightarrow$ *2.* readily follows. Then *2.* $\Rightarrow$ *3.* is immediate. To see *3.* $\Rightarrow$ *4.*, denote

$$S_t^\omega(x, y) \equiv \mathbb{E}\big[S_t(x, y)\big| \cap_{F \in \mathcal{F}_t : \omega \in F} F\big]$$

a particular realization of the ex-ante surplus. Suppose by contradiction that no such $\hat{y}_t(x)$ can be chosen. Then, in particular, there exist $x_2 > x_1$ and $\omega \in \Omega$ such that $y_1 \in \arg\max_y S_t^\omega(x_1, y)$, $y_2 \in \arg\max_y S_t^\omega(x_2, y)$, yet $y_1 > y_2$. Then $S_t(x_1, y_1) \geq S_t(x_1, y_2)$. Hence single-crossing implies that

$$S_t^\omega(x_1, y_1) \geq S_t^\omega(x_1, y_2) \quad \Rightarrow \quad S_t^\omega(x_2, y_1) > S_t^\omega(x_2, y_2).$$

But then y_2 cannot be the ex-ante strictly preferred partner's type of agent type x_2, which is absurd. □

This result demonstrates that supermodularity is a first-order reason for positive assortative matching, even in the presence of search frictions, aggregate uncertainty, and ex-post heterogeneity.

4.2 Single-peaked preferences

We now turn our attention to single-peaked preferences. Economically speaking, this captures the idea that each agent type x has an ex-ante preferred type such that agents more closely resembling her preferred type $y_t(x)$ are preferred over those agent types that do so to a lesser degree. To introduce this notion formally, we make use of the finite differences operator: define $\triangle_y h(x, y) \equiv h(x, y_+) - h(x, y)$ for arbitrary function h. Here y_+ denotes the smallest type $y \in Y$ strictly greater than y. (Similarly, denote y_- the greatest type $y \in Y$ strictly smaller than y.)

[13] A similar result has been reported in the working paper by Shimer and Smith (2000) at the steady state.

Definition 3 (Single-peaked)**.** *$y \mapsto S_t(x,y)$ is single-peaked if there exists $\hat{y}_t(x)$ such that $\triangle_y S_t(x,y)$ is strictly positive for all $y < \hat{y}_t(x)$ and strictly negative for all $y > \hat{y}_t(x)$.*

In the same vain, we say that a function is weakly single-peaked if the preceding holds for weakly positive and weakly negative differences. Finally, throughout refer to $y_t(x) = \arg\max_y S_t(x,y)$ and $x_t(y) = \arg\max_x S_t(x,y)$ as the correspondences giving the preferred partner's types. if preferences are (weakly) single-peaked, then those correspondences are convex-valued.

Single-peaked preferences are a much more demanding property to satisfy than single-crossing. As already observed by Shimer and Smith (2000), a given type x may express a desire to match with high and low types y, but not so with intermediate types, even when ex-ante match output is supermodular.[14] As a keystone of their analysis, Shimer and Smith (2000) report sufficient conditions on f under which single-peaked preferences obtain nonetheless at the steady state. In this subsection, we prove that the sufficiency of those conditions extends to the richer environment studied in this paper.

To begin with, we introduce two conditions on ex-ante output f which ensure that the ex-ante surplus is singled-peaked. To simplify the interpretation of the conditions, we express them in terms of differences, $\triangle_y f$.[15]

Definition 4. *$\triangle_y f(x,y)$ is log supermodular if for any $x_1 < x_2$ and $y_1 < y_2$ we have*

$$\frac{\triangle_y f(x_2,y_2)}{\triangle_y f(x_2,y_1)} \geq \frac{\triangle_y f(x_1,y_2)}{\triangle_y f(x_1,y_1)}.$$

If both $\triangle_y f(x,y)$ and $\triangle_x f(x,y)$ are log supermodular, we write that $\triangle f(x,y)$ is log supermodular.

Definition 5. *A function $\triangle_y f(x,y)$ is log supermodular in differences if for any $x_1 < x_2 < x_3$ and $y_1 < y_2$ we have*

$$\frac{\triangle_y f(x_3,y_2) - \triangle_y f(x_2,y_2)}{\triangle_y f(x_2,y_2) - \triangle_y f(x_1,y_2)} \geq \frac{\triangle_y f(x_3,y_1) - \triangle_y f(x_2,y_1)}{\triangle_y f(x_2,y_1) - \triangle_y f(x_1,y_1)}.$$

Observe that if $\triangle_y f(x,y)$ is log supermodular in differences, then so is $\triangle_x f(x,y)$ (with the roles of x and y) reversed. We thus simply write that $\triangle f(x,y)$ is log supermodular in differences.

These conditions ensure that the degree of complementarity in output is sufficiently high. First note that log supermodularity is a condition on the levels of the difference in ex-ante output, whereas log supermodularity in differences is a condition on the curvature of the difference in ex-ante output. Specifically, it asserts that under log supermodularity in differences $y \mapsto \triangle_x f(x_2,y)$ is more convex than $y \mapsto \triangle_x f(x_1,y)$, which admits an interpretation in terms of risk preferences that we draw on subsequently. It follows that neither of the two conditions implies the other. Provided that f is differentiable with strictly positive derivatives, these conditions are equivalent to saying that $\partial_x f$ and $\partial^2_{xy} f(x,y)$ are log supermodular. The most natural ex-ante output which satisfies these conditions is $f(x,y) = xy$ (as first explored by Lu and McAffee (1996)).

[14] They refer to this property as convexity of matching sets, but their proof relies on establishing that ex-ante match output is quasi-concave, i.e., agents' preference relations are single-peaked.

[15] We say that these conditions hold strictly if the defining equation is strict across all types.

Theorem 2 (single-peaked preferences)**.** *Suppose that ex-ante output f is supermodular and $\triangle f$ is log supermodular and log supermodular in differences, with at least one holding strictly. Then $y \mapsto S_t(x, y)$ is single-peaked.*

The ansatz of the proof proceeds by contraposition: suppose preferences were not single-peaked. Then there exists a type x and three adjacent agent types $y_1 < y_2 < y_3$ such that both $y_1 \succsim_t^x y_2$ and $y_3 \succsim_t^x y_2$, which is equivalent to $S_t(x, y_1) > S_t(x, y_2)$ yet $S_t(x, y_3) > S_t(x, y_2)$, or

$$\triangle_y V_t(y_1) > \triangle_y f(x, y_1) \quad \text{yet} \quad \triangle_y V_t(y_3) < \triangle_y f(x, y_3). \tag{5}$$

At this stage, it is unclear how to proceed, for the difference in the value of search $\triangle_y V_t(y)$ is an obscure object. It is only implicitly characterized by a system of FBSDEs which usually does not admit a closed-form solution. We will next present two results, the mimicking argument and a comparison of lotteries, which lead to a contradiction of the alleged violation of single-peaked preferences, namely condition 5.

Step 1: the mimicking argument

The main ingenuity of our proof is to provide tight bounds on the difference in the value of search between two agents, $\triangle_y V_t(y)$, solely in terms of the ex-ante match-output. Those are formalized in the following lemma that we refer to as the *mimicking argument.* It is reminiscent of the mimicking argument developed by Bonneton and Sandmann (2019) for the non-transferable utility paradigm.

Lemma 3 (mimicking argument)**.** *For every $x_1 \in X$ and time t there exist non-negative random weights on Y, denoted $Q_t(y|x_1) : \mathcal{F}_t \to \mathbb{R}_+$ with $\sum_{y \in Y} Q_t(y|x_1) \leq 1$ such that for every $x_2 \in X$ the following inequality obtains in equilibrium:*

$$V_t(x_2) - V_t(x_1) \geq \sum_{y \in Y} \big[f(x_2, y) - f(x_1, y)\big] Q_t(y|x_1).$$

In words, the lemma states that the difference in the value of search between any two agents can be bounded by a weighted sum over the difference in match-output. Crucially, holding agent type x_1 fixed, these weights hold true for any other type x_2.

The proof of this result relies on the efficiency of Nash bargaining as asserted by lemma 1. This lemma states that the value of search dominates the discounted expected match payoff one would obtain by adopting any other acceptance threshold θ, i.e., $V_t(\cdot) \geq W_t^{\theta}(\cdot)$. In particular, this affords a lower bound on a given type x_2's value of search, where x_2 "mimics" the equilibrium acceptance threshold of agent type x_1. Formally, set $\theta_t(x_2, \cdot) = \theta_t^*(x_1, \cdot)$ for all $t \in [0, T]$. In doing so, agent type x_2 exactly replicates agent type x_1's matching probabilities. Finally, subtracting $V_t(x_1)$ from $V_t(x_2) \geq W_t^{\theta}(x_2)$ gives

$$V_t(x_2) - V_t(x_1) \geq \mathbb{E}\Big[\int_t^T e^{-\rho(\tau - t)} \sum_{y \in Y} \big[\pi_\tau^{\theta}(x_2, y) - \pi_\tau(x_1, y)\big] \underbrace{P_t(d\tau)(y|x_1)}_{x_1\text{'s matching probability}} \big| \mathcal{F}_t\Big].$$

When two agent types adopt identical acceptance thresholds, they match conditional on identical realizations of production shocks ξ. Since those are identically distributed across agents, the difference in match payoffs admits further simplification:

$$\pi^{\theta}_{\tau}(x_2, y) - \pi_{\tau}(x_1, y) = (1 - \alpha^X_{\tau})\big(V_{\tau}(x_2) - V_{\tau}(x_1)\big) + \alpha^X_{\tau}\big(f(x_2, y) - f(x_1, y)\big).$$

Substituting back into the preceding, we get

$$\begin{aligned} V_t(x_2) - V_t(x_1) \geq \sum_{y \in Y} \left[f(x_2, y) - f(x_1, y)\right] \mathbb{E}\Big[\int_t^T e^{-\rho(\tau - t)} \alpha^X_{\tau} P_t(d\tau)(y|x_1) \Big| \mathcal{F}_t\Big] \\ + \mathbb{E}\Big[\int_t^T \big(V_{\tau}(x_2) - V_{\tau}(x_1)\big) e^{-\rho(\tau - t)} (1 - \alpha^X_{\tau}) \bar{P}_t(d\tau)(x_1) \Big| \mathcal{F}_t\Big]. \end{aligned}$$

The first term is, as desired, a weighted sum over the difference in match output, the second term is (a weighted discounted future average of) the expected difference in value of search. Note that if the difference in value of search, $V_{\tau}(x_2) - V_{\tau}(x_1)$, did not fluctuate over time, rearranging the inequality readily gives the desired bound.

The remainder of the proof proceeds by iteration. First, when applying the above reasoning to (the weighted discounted future average of) $V_{\tau}(x_2) - V_{\tau}(x_1)$, we once more obtain two terms: the first term is, as desired, a weighted sum over the difference in match output, the second term is (a weighted discounted future average over averages of) the expected difference in value of search. After k iterations of this reasoning, we obtain $k+1$ terms, the first k of which converge to a geometric series of the desired weighted sum over the difference in match output, and the $k+1$th term is shown to converge to zero as k grows large - which allows us to conclude the proof.

The *Matryoshka dolls* offer an intuitive representation of our iterative process. The Matryoshkas are a countable number of dolls, one wrapped into another. In our proof a doll corresponds to the difference in value of search (an object we do not understand), whereas its shell corresponds to a weighted sum over the difference in match output (an object we do understand). As we open the first doll, we keep the shell and find inside another doll, inside of which there is another doll. As iterations go on, we accumulate shells. In the meantime, dolls become exceedingly small, which eventually allows us to get rid of them altogether.

We find it worthwhile to mention that in Bonneton and Sandmann (2019) we had presented a different mimicking argument. In both papers, the underlying idea is to let one agent mimic someone else in order to facilitate the comparison of the value of search across types. In the NTU paradigm, the result hinges on *payoff monotonicity.* Superior types, being more desirable, can exploit their superior match offerings and replicate match outcomes of any inferior type. In this paper, the result obtains for a different reason, namely the *efficiency of the Nash bargaining sharing rule.* The ensuing result is somewhat stronger: an agent can mimic not only inferior types, but also superior ones.[16]

[16] In the NTU paradigm, the desired bound straightforwardly obtains from one agent mimicking someone else. This is due to the independence between payoffs and the value of search. Here we use an iterative argument to deal with such dependence.

Step 2: comparison of lotteries

With the mimicking argument at hand, we can reformulate the negation of singled-peaked preferences (5) as a choice over lotteries:

$$\sum_{z\in X} \triangle_y f(z,y_1)Q_t(z|y_2) > \triangle_y f(x,y_1) \quad \text{yet} \quad \sum_{z\in X} f(z,y_3)Q_t(z|y_2) < \triangle_y f(x,y_3). \tag{6}$$

To give this condition a figurative meaning, take $\triangle_y f(x,y)$ to be a (fictitious) utility of agent type y over partners x. Under this figurative meaning, condition (6) means that the greater agent type y_3 prefers the certain partner x over an uncertain partner z (given by weights $Q_t(\cdot|y_2)$), whereas the lower agent type y_1 prefers the uncertain partner z instead. To make use of the theory of risk and draw on a beautiful theorem by Pratt (1964), we seek to normalize the weights $Q_t(\cdot|y_2)$ such that they correspond to a lottery, i.e., sum to one. Pratt (1964) shows that *given arbitrary $y_3 > y_2$ the following statements are equivalent:*

1. *Agent type y_3 is more risk-averse than agent type y_1; that is, y_3 does not accept a lottery that is rejected by y_1;*
2. *$x \mapsto \triangle_y f(x,y)$ is log supermodular in differences.*

In the appendix we present detailed normalization arguments drawing on supermodularity of f and log supermodularity of $\triangle f$ such that, indeed, (6) obtains for some lottery, i.e., $x \mapsto Q_t(\cdot|y_2)$ sums to one. Then for agent type y_1 not to prefer the lottery $Q_t(\cdot|y_2)$ that agent type y_2 rejected, it suffices to require that $\triangle_y f(x,y)$ is log supermodular in differences.

Shimer and Smith (2000) draw on the same connection with the theory of risk. They prove that identical conditions, supermodularity of f and log supermodularity and log supermodularity in differences of $\triangle f$, ensure single-peaked preferences at the steady state.

Whether those conditions are necessary is not resolved. It is an open question that shall be tackled in future research.

Although the ex-ante surplus, the matching probabilities, and the expected payoffs are single-peaked and satisfy single-crossing, the equilibrium transfers do not inherit these good proprieties; and the distribution over wages depends heavily on the distribution of pair-specific shocks (as shown in appendix C.3).

5 Probabilistic Assortative Matching

In this section we characterize aggregate matching patterns by drawing on the individual preferences studied in the preceding section. We then discuss when and why matching patterns exhibit PAM and NAM.

5.1 Definition of probabilistic assortative matching

Positive assortative matching is the intuitive notion that agents of similar characteristics tend to match with one another. However intuitive, the mathematical definition which

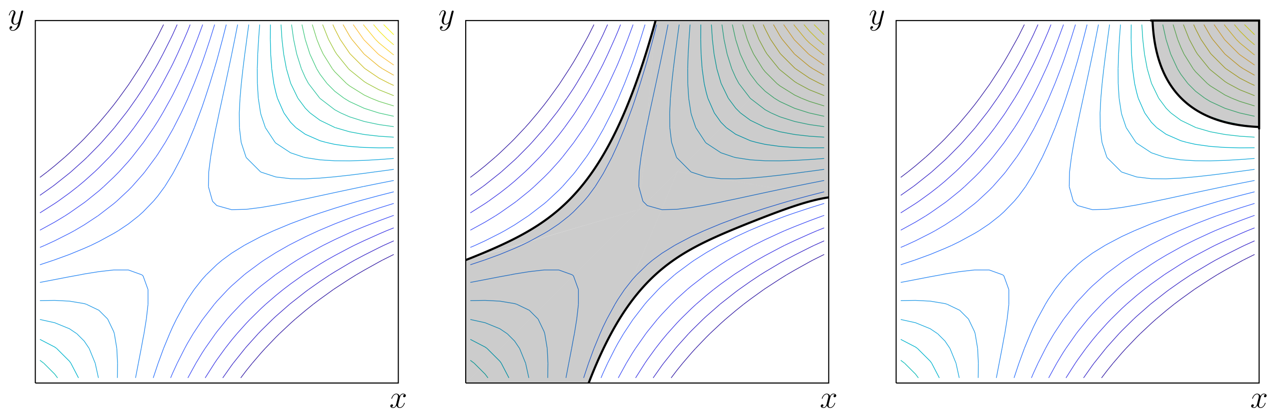

Figure 3: Not every upper counter matching set is a lattice ⇔ Not all iso-lines ↗.

corresponds to this notion is neither self-evident nor unique in a model that generates probabilistic matching patterns as we observe in the data.[17]

In this paper we are interested in agents' choices. Concurrently, we take positive assortative matching to be a property of preferences $\succsim_t^x$ and $\succsim_t^y$. In our work the probabilistic matching function $y \mapsto m_t(x, y)$ is a utility representation of preferences $\succsim_t^x$. It is thus without loss of generality to cast our definition of PAM in terms of *ordinal* properties of the probabilistic matching function $m_t(x, y)$. This function can best be visualized by a time-moving heat map over pairs of types x and y where lighter colors correspond to greater match probabilities.[18]

The predominant definition of PAM within the realm of random search is due to Shimer and Smith (2000). Our definition is a generalization of theirs to environments in which match outcomes are probabilistic, not binary. Motivated by their work, we represent agents' joint decisions $m_t(x, y)$ by upper contour matching sets, that is, all those pairs of types (x, y) which match with probability weakly greater than some given p:

$$U_t(p) = \big\{(x, y) : m_t(x, y) \geq p\big\}.$$

What Shimer and Smith (2000) require on $U_t(1)$ for positive assortative matching to obtain, we require on $U_t(p)$ for all p. In particular, we require as a necessary condition for PAM that upper contour matching sets $U_t(p)$ be a lattice *for all* $p \in [0, 1]$.

Definition 6. *The upper contour matching set $U_t(p)$ is a lattice if for any types $x_1 < x_2$ and $y_1 < y_2$, the upper contour matching set $U_t(p)$ is such that*

$$(x_1, y_2) \in U_t(p) \text{ and } (x_2, y_1) \in U_t(p) \quad \Rightarrow \quad (x_1, y_1) \in U_t(p) \text{ and } (x_2, y_2) \in U_t(p).$$

Take four types $x_1 < x_2$ and $y_1 < y_2$, such that both non-assortative pairs, (x_1, y_2) and (x_2, x_1), match with probability greater than p. Then the lattice property ensures that both assortative pairs, (x_1, y_1) and (x_2, y_2), equally match with probability greater than p.

[17] In the frictionless environment proposed by Becker (1973) said notion admits a unique formal definition: every agent matches with a unique agent type which is increasing in his own type.

[18] Another choice of domain of the definition is the flow rate of match creation, i.e. $m_t(x, y)\lambda_t(y)\mu_t(x)$, which we observe in the data. We choose not to do so because we believe it is important to disentangle choices, m_t, from the physical environment, μ_t and λ_t.

As a second necessary condition for PAM we require that individual upper contour matching sets be convex.[19] Formally, the individual upper contour set, defined as $U_t(x;p) = \{y : m_t(x,y) \geq p\}$, is convex if for any three agent types $y_1 < y_2 < y_3$, $y_1 \in U_t(x;p)$ and $y_3 \in U_t(x;p)$ imply that $y_2 \in U_t(x;p)$. This conveys the idea that the greatest match probabilities of any fixed agent type x are concentrated around an interval of types y, the ex-ante ideal partners for type x. The more a type differs from the ideal partners y, the lower is the probability of matching.

To summarize, we generalize the definition of PAM provided by Shimer and Smith (2000) as follows:

Definition 7 (PAM)**.** *There is probabilistic positive assortative matching (PAM) if*

(i) *the upper contour matching set $U_t(p)$ is a lattice,*

(ii) *individual upper contour matching sets $U_t(x;p)$ are convex for all agent types,*

for all $p \in [0,1]$.

We wish to emphasize that this definition is about ordinal properties of the preference relations $\succsim_t^x$, not its particular representation: a given utility representation of $\succsim_t^x$ satisfies the PAM-defining properties if and only if every other representation satisfies those properties. In particular, we could equivalently show that all upper contour sets of the ex-ante surplus, $S_t(x,y)$, are a lattice and satisfy individual convexity. We chose to state the definition using the probabilistic matching function, rather than the preference relation, as to make explicit the connection between Shimer and Smith's definition and ours.[20] In the subsequent section, we show that PAM can equivalently be stated in terms of intuitive conditions on the preference relations $\succsim_t^x$.

For illustrative purposes, it is convenient to visualize PAM by drawing on lower and upper level lines: $l_t(x;p) = \min\{y : m_t(x,y) \geq p\}$ and $u_t(x;p) = \max\{y : m_t(x,y) \geq p\}$, where $l_t(x,p)$ ($u_t(x;p)$) is the smallest (highest) type y with who agent type x matches with probability greater than p. As shown in proposition in the appendix, provided that individual counter sets are convex, there is PAM if and only if $x \mapsto l_t(x;p)$ and $x \mapsto u_t(x;p)$ are non-decreasing for all p. This is depicted in figure **??**. One can check visually that the upper contour matching set from figure 3 (center) is a lattice as both upper and lower level lines are increasing. However, the upper contour matching set in figure 3 (right) is not. The lattice property is upset for precisely those agent types for which level lines are

[19]In Shimer and Smith (2000), the definition of PAM tacitly assumes convexity. Indeed, the convexity of $U_t(x;1)$ is implied by the lattice property in their framework. Here is a sketch of the argument: in a stationary environment with binary matching probabilities, $U_t(x,1)$ is necessarily non-empty (for otherwise the agent searches forever). More generally, when the individual upper contour matching set $U_t(x,p)$ is non-empty for every type x, then $U_t(p)$ being a lattice implies that $U_t(p)$ is convex (proof in proposition 4 in the appendix). In non-stationary and/or probabilistic environment, individual upper contour sets can be non-empty; the lattice property does not imply convexity of individual upper contour matching sets.

[20]In contrast to our definition, Shimer and Smith do not cast their definition in terms of preferences. If one were to introduce an identical preference relation in their framework as we do, i.e. a ranking over the preferred partner's type that one would like to meet, their definition of PAM would not be an ordinal property of said preference relation. And their binary $m_t(x,y)$ as opposed to their $S_t(x,y)$ would not be a utility representation of said preference relation. This follows from the simple fact that their definition of PAM bears on properties of a single upper counter set, $\{y : S_t(x,y) \geq 0\}$. In order to characterize ordinal properties of a preference relation via upper counter sets, we know from consumer choice theory that imposed properties must be true for all such sets.

decreasing. In particular, the heat map depicts an instance in time where PAM is not satisfied.

5.2 Preferences for assortative matching

In section 4 we had studied single-peaked preferences and single-crossing. As it turns out, these two properties, albeit necessary, do not suffice to establish PAM. The reason is that both are comparative properties regarding preferences *within* a given population: single-crossing is concerned with how a lower type x_1's preferences compare to a higher type x_2's preferences; single-peaked preferences are concerned with preferences of a single agent type. The lattice property, in contrast, is a property that considers both populations jointly. It bears on how preferences *across* populations relate to one another.[21] In order to provide an equivalent preference representation of PAM, we need not only weakly single-peaked preferences and weak single-crossing, but also a condition that ties preferences across the two populations. This condition, which to the best of our knowledge is new to the matching literature, is *reciprocity* of matching preferences.

Definition 8 (reciprocity)**.** *We say that $\succsim_t^x$ and $\succsim_t^y$ as represented by $y \mapsto m_t(x, y)$ and $x \mapsto m_t(x, y)$ exhibit reciprocity if for arbitrary types $(x, y) \in X \times Y$*

$$y < \underline{y}_t(x) \ \Rightarrow\ \underline{x}_t(y) \leq x \quad \text{and} \quad y > \overline{y}_t(x) \ \Rightarrow\ \overline{x}_t(y) \geq x,$$

where $\underline{y}_t(x) = \min \arg\max_y m_t(x, y)$ and $\overline{y}_t(x) = \max \arg\max_y m_t(x, y)$; likewise for $\underline{x}_t(y)$ and $\overline{x}_t(y)$.

Interchangeably, we say that a function is reciprocal if preferences represented by said function are reciprocal. This definition is necessarily tangled, for it must grapple with the subtleties of indifference between several agent types. In words, reciprocity means: if a given agent type y is inferior to an agent type x's set of preferred types, then such type y must have preference to meet someone inferior to type x; likewise, if a given agent type y is superior to an agent type x's set of preferred types, then such type y must have a preference to meet someone superior to type x.

Having introduced all the pieces, we can now make precise the relationships between introduced primitives on preferences and the constituent parts of the definition of PAM. We begin by stating the obvious:

Remark 4. *$\succsim_t^x$ is weakly single-peaked if and only if $U_t(x; p)$ is convex for all $p \in [0, 1]$.*

The link between preferences and PAM is established by the following proposition:

[21] A different way of making this point is to say that single-peaked preferences and single-crossing are *ordinal* properties of an individual agent's utility. The lattice property, while being an ordinal property of joint utilities as discussed in the preceding subsection, can not be cast as an ordinal property of an individual agent's utility: take arbitrary utility representations $u_t^x(y)$ of preference relation $\succsim_t^x$, e.g. let $u_t^x(y) \equiv m_t(x, y)$, and take arbitrary monotone transformations T^x *for each* x. Then $T^x \circ u_t^x(y)$ is another representation of preference relation $\succsim_t^x$ which preserves single-peaked preferences and single-crossing from the viewpoint of agent types x (but not necessarily from the viewpoint of agent types y): $y \mapsto T^x \circ u_t^x(y)$ is single-peaked and $T^{x_1} \circ u_t^{x_1}(y_2) \geq T^{x_1} \circ u_t^{x_1}(y_1)$ implies that $T^{x_2} \circ u_t^{x_2}(y_2) > T^{x_2} \circ u_t^{x_2}(y_1)$. Meanwhile, the lattice property of upper contour matching sets $U_t(p)$ induced by some $m_t(x, y)$ is not preserved under x-contingent monotone transformations of $y \mapsto m_t(x, y)$.

Proposition 2. *Suppose that upper contour sets $U_t(p)$ are a lattice for all p. Then preferences $\succsim_t^x$ and $\succsim_t^y$ satisfy reciprocity and weak single-crossing. Suppose that preferences $\succsim_t^x$ and $\succsim_t^y$ satisfy reciprocity, weak single-crossing and are weakly single-peaked. Then upper contour sets $U_t(p)$ are a lattice for all p.*

The key insight from this subsection is then that PAM admits an equivalent representation, expressed in terms of primitives of preferences:

Corollary 2 (Preference representation result of PAM)**.** *There is PAM if and only if preferences are weakly single-peaked, and satisfy reciprocity and weak single-crossing.*

Single-peaked preference is the main sufficient condition for PAM in Shimer and Smith (2000), as it implies that individual matching set $U_t(x, 1)$ are convex. But in their context, it is not necessary, as single-peaked preference is not implied by the convexity of individuals matching set. In Atakan (2006), individual matching sets are proven to be convex when the match output is supermodular, but supermodularity alone does not give single-peaked preference.

Lack of reciprocity

Reciprocity owes its name to the fact that, provided that preferences are also weakly single-peaked and satisfy weak single-crossing, preferences over the preferred partner's type are reciprocated: every agent type x's set of preferred partner's type is at the very least adjacent to some agent type y for whom x is a preferred partner. To formalize this claim, define $\mathcal{P}_t$ the set of pairs of types where at least one type is the other's preferred partner's type:

$$\mathcal{P}_t \equiv \{(x, y) : x \in x_t(y) \text{ or } y \in y_t(x)\}.$$

We establish that PAM has the following, implausibly strong, property:

Proposition 3. *Suppose there is PAM. Then,*

- *$\mathcal{P}_t$ is convex in the x and the y dimension, i.e., for $x_1 < x_2 < x_3$, if (x_1, y) and (x_3, y) are in $\mathcal{P}_t$, then so is (x_2, y), likewise for types y,*
- *$\mathcal{P}_t$ is path-connected along the horizontal, vertical, or diagonal,*[22]
- *$(x_{\min}, y_{\min})$ and $(x_{\max}, y_{\max})$ are in $\mathcal{P}_t$.*

Visually, this property means that matching patterns, as depicted by a heat-map, can be likened to a mountain ridge. This ridge ranges from the pair composed of the lowest types, $(x_{\min}, y_{\min})$, to the pair composed of the highest types, $(x_{\max}, y_{\max})$. None of the three conditions is generically satisfied when there are single-peaked preferences that satisfy single-crossing, but not reciprocity.

We emphasize that the lack of reciprocal preferences, and thus PAM, arises generically for any $f(x, y)$ such that the preferences represented by $f(x, y)$, i.e., $y \mapsto f(x, y)$ and $x \mapsto$

[22]Formally, for any two $(\underline{x}, \underline{y})$ and $(\overline{x}, \overline{y})$ there exists a path $((\underline{x}, \underline{y}), (x_1, y_1), ..., (x_K, y_K), (\overline{x}, \overline{y}))$ such that if (x', y') succeeds (x, y), then $(x', y') \in \{(x_-, y_-), (x_-, y), (x_+, y), (x_+, y_+)\}$, where y_+ denotes the smallest type $y \in Y$ strictly greater than y. And, Here y_- denotes the smallest type $y \in Y$ strictly greater than y.

$f(x, y)$, do not satisfy reciprocity; that is, reciprocity of $f(x, y)$ is a necessary condition for PAM. This follows from the simple observation that when search frictions are sizeable and terminal values are zero, ex-ante surplus $S_t(x, y)$ is approximately equal to ex-ante match output $f(x, y)$. Thereby preferences over meetings are approximately given by those represented by $f(x, y)$.

Does reciprocity of $f(x, y)$, conversely, imply that preferences over meetings (as represented by $S_t(x, y)$ or $m_t(x, y)$) are reciprocal? In appendix D.6, we provide a counter-example which revolves around a symmetric population comprising three types; $y \mapsto f(x, y)$ is maximal for complementary types $y = x$ and $f(x, y)$ is supermodular (but does not satisfy log supermodularity in differences). We show that there exist economic fundamentals for which reciprocity does not obtain: low and high types exhibit self-preferences, whereas the intermediate type prefers to match with the lowest type. This shows that reciprocity of preferences over meetings is difficult to satisfy. We believe, however, that reciprocity of $f(x, y)$ and preceding complementarity conditions taken together are sufficient for reciprocity of preferences over meetings—and PAM.

In what follows we establish sufficiency for the last item, self-preferences of boundary types. It will play an important role in generalizing Shimer and Smith (2000)'s sorting result.

Lemma 4 (Boundary conditions)**.** *Suppose that output f is supermodular. If $y \mapsto f(x_{\max}, y)$ is maximal for $y_{\max}$, then $\overline{y}_t(x_{\max}) = y_{\max}$; likewise, if $y \mapsto f(x_{\min}, y)$ is maximal for $y_{\min}$, then $\underline{y}_t(x_{\min}) = y_{\min}$.*

We stress that $y \mapsto f(x, y)$ non-increasing is a strong assumption as it rules out strict vertical differentiation of types. As a by-product of our analysis, we show this conditions, albeit sufficient is not necessary and can be weakened at the steady state. We prove that log supermodularity of f, as implied by their conditions under vertically differentiated types, renders this boundary condition obsolete (refer to appendix D.5 for the proof).

5.3 Main result

Our main result provides a comparative characterization of matching patterns when preferences are weakly single-peaked and satisfy single-crossing, but need not satisfy reciprocity. Those properties arise when f is supermodular and $\triangle f$ is log supermodular and log supermodular in differences, with at least one holding strictly (as established by theorem 2 and proposition 1). To that end, we partition the set of pairs of types along the preferred partner's type correspondence, and show how match probabilities $m_t(x, y)$ compare across pairs of types. As a corollary to our comparative characterization of matching patterns, we can determine those pairs of types for whom our theoretical definition of PAM obtains, and those for whom NAM (i.e., PAM upon reordering of types X) obtains.

To proceed, we partition the space of pairs $X \times Y$ into four regions, which partially overlap at the graphs of the preferred partner's types' correspondences $x_t(y)$ and $y_t(x)$, i.e., $\mathcal{P}_t$. We call the upper and lower enclosure of $\mathcal{P}_t$ those sets of pairs of types that lie in between the two correspondences:

$$\overline{\mathcal{E}}_t \equiv \{(x, y) : y \leq \underline{y}_t(x) \wedge x \leq \underline{x}_t(y)\} \quad \text{and} \quad \underline{\mathcal{E}}_t \equiv \{(x, y) : y \geq \overline{y}_t(x) \wedge x \geq \overline{x}_t(y)\}.$$

Similarly we define its complements, the set above and below those correspondences the upper and lower outer set:

$$\underline{\mathcal{O}}_t \equiv \{(x,y) : y \leq \overline{y}_t(x) \wedge x \geq \underline{x}_t(y)\} \quad \text{and} \quad \overline{\mathcal{O}}_t \equiv \{(x,y) : y \geq \underline{y}_t(x) \wedge x \leq \overline{x}_t(y)\}.$$

Theorem 3 (characterization of matching patterns)**.** *Suppose that preferences are weakly single-peaked. Then,*

- *match probabilities for pairs in $\underline{\mathcal{E}}_t$ rise for lower x and lower y, i.e.,* $\leftarrow\downarrow$,
- *match probabilities for pairs in $\overline{\mathcal{E}}_t$ rise for higher x and higher y, i.e.,* $\uparrow\rightarrow$,
- *match probabilities for pairs in $\overline{\mathcal{O}}_t$ rise for higher x and lower y, i.e.,* $\downarrow\rightarrow$,
- *match probabilities for pairs in $\underline{\mathcal{O}}_t$ rise for lower x and higher y, i.e.,* $\leftarrow\uparrow$.

Suppose further that preferences satisfy single-crossing. Then both the upper and lower enclosure $\overline{\mathcal{E}}_t, \underline{\mathcal{E}}_t$ and the upper and lower outer set $\overline{\mathcal{O}}_t, \underline{\mathcal{O}}_t$ are a lattice.

Proof. The first statement is an immediate consequence of single-peaked preferences, of which $y \mapsto m_t(x,y)$ and $x \mapsto m_t(x,y)$ are representations. To see that the enclosures and outer sets are a lattice, observe that proposition 5 establishes an identical result: provided that $U_t(x;p)$ is convex, $U_t(p)$ is a lattice if and only if the p-level lines (which bound $U_t(p)$) are non-decreasing. Here, both enclosures and upper outer sets are non-empty by construction, convex due to single-peaked preferences and bounded by the lower and upper bounds of the preferred partner's type correspondences, $y \mapsto \underline{x}_t(y), \overline{x}_t(y)$ and $x \mapsto \underline{y}_t(x), \overline{y}_t(x)$. Proposition 1, item 4 establishes that those are non-decreasing, hence the result. □

A characterization of matching patterns in terms of PAM and NAM readily follows from theorem 3. NAM is converse notion of PAM; there is NAM if by reversing the order of types for one side of the population, PAM obtains. Or, NAM arises if, given four types $x_1 < x_2$ and $y_1 < y_2$, the non-assortative pairs (x_1, y_2) and (x_2, y_1) have weakly greater match probability than at least one of the assortative pairs.

Corollary 3 (Equilibrium sorting)**.** *Under theorem 3's assumptions, there is PAM in the outer sets $\overline{\mathcal{O}}_t$ and $\underline{\mathcal{O}}_t$, and NAM in the enclosure $\overline{\mathcal{E}}_t$ and $\underline{\mathcal{E}}_t$.*

In the introduction we ascribed our results to a tension between two conflicting forces, type complementarity, i.e., $f(x,y)$ is supermodular, and vertical differentiation of types, i.e., $f(x,y)$ is increasing. We explored a channel through which search frictions give rise to unanimous preferences for superior types. If the value of search is low for everyone, then everyone will want to meet those agents with whom they produce the greatest ex-ante match output, not their complementary type. We wish to emphasize, however, that agents may likewise deviate and prefer inferior types over their complementary type—even when types are vertically differentiated: if the value of search is high for greater and low for inferior types, the loss in output of matching with inferior types may be outweighed by the lesser compensation those inferior types demand.[23]

[23] To obtain this, we can just consider the counter-example in appendix D.6, but change the payoff matrix. Now, $f(x,y) = xy$ and $q = 0.8$, $x_3 = 1$, $x_2 = 0.5$, $x_1 = 0.4$, we obtain the following: $V(x_2) = 0.4$, $V(x_2) = 0.07$, $V(x_1) = 0$, $S(x_2, x_2) = 0.11$, $S(x_1, x_2) = 0.13$, $S(x_3, x_2) = 0.03$. So the medium type prefers to match with the lowest type.

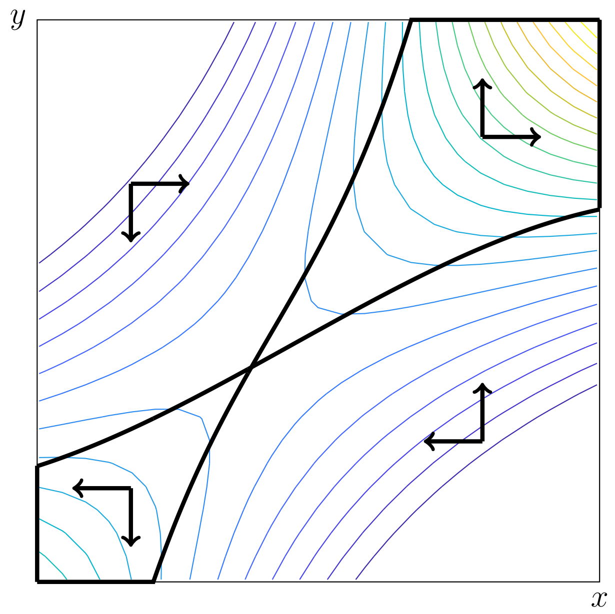

Figure 4: Illustration of theorem 2: Single-crossing and single-peaked preferences

We caution that our main result should not be interpreted as a quantitative assessment. To us, PAM and NAM are ordinal properties. Their study gives rise to an ordinal characterization of sorting patterns that is robust to changes in the physical environment that do not concern $f(x, y)$. From an empirical perspective this is desirable, because it avoids mis-attributing changes in the frequency of matches of given characteristics x and y to changes in match complementarity $f(x, y)$.

5.4 Revisiting Shimer and Smith (2000)

The main contribution of Shimer and Smith (2000) is to identify sufficient conditions on $f(x, y)$ for which PAM obtains in the steady state and without pair-specific production shocks. Their set of conditions include the strong complementarity conditions required in theorem 2 and the boundary conditions from lemma 4 (which prohibit vertically differentiated types).

Here we show that a version of Shimer and Smith's original result can be recovered in our more general environment under similar conditions as those imposed in their paper. We emphasize that those conditions are notably stronger than the assumptions made so far: we require their boundary conditions, and we also have to endow the type space with a distance and impose continuity of match output.

To proceed, let $\hat{p}_t$ the min max-probability, the smallest of all agent's maximal match probabilities:

$$\hat{p}_t \equiv \min_x \max_y m_t(x, y)$$

Observe that, equivalently, $\hat{p}_t = \min_{(x,y)\in\mathcal{P}_t} m_t(x, y)$. Since $\mathcal{P}_t \subseteq \underline{\mathcal{E}}_t \cup \overline{\mathcal{E}}_t$, this probability is greater than the the smallest match probability in the enclosure of pairs (where PAM is upset):

$$p_t^e \equiv \min_{(x,y)\in \underline{\mathcal{E}}_t \cup \overline{\mathcal{E}}_t} m_t(x,y),$$

i.e., $\hat{p}_t \geq p_t^e$. p_t^e gives a threshold probability up to which upper counter matching sets do satisfy the PAM-defining probabilities. Indeed, under the assumptions of theorem 3, for all $p \leq p_t^e$, upper counter sets satisfy the PAM-defining properties. To reduce the significance of NAM, we seek to infer from large maximal match probabilities $\hat{p}_t$, that also p_t^e is large. If so, then (approximately) there is PAM. It turns out that this assertion requires two properties: first we need to endow the type space with a distance and require continuity of match output. Thus, denote d a distance on $X \times Y$. Secondly, we need to impose boundary conditions on preferences.

Assumption 2. *Assume that preferences satisfy the following:*

- *there are weakly single-peaked preferences that satisfy single-crossing;*
- $(x_{\min}, y_{\min}), (x_{\max}, y_{\max}) \in \mathcal{P}_t$

Moreover, we assume that output is uniformly continuous and Ξ_t is Lipschitz continuous:

- *the distance between horizontally or vertically adjacent types is less than $\frac{r}{\underline{N}}$ where $\underline{N} \equiv \min\{N^X, N^Y\}$; $f(x,y)$ is Lipschitz-continuous with constant $\frac{q}{r}$;*
- *Ξ_t is Lipschitz-continuous with constant $\frac{L^\xi}{3q}$.*

As in Shimer and Smith, boundary conditions orient matching sets. Here this means that $\mathcal{P}_t$ encloses the NAM-region $\underline{\mathcal{E}}_t \cup \overline{\mathcal{E}}_t$ both from above and below. As shown in lemma 4, self-preference at the boundary, i.e., $(x_{\min}, y_{\min}), (x_{\max}, y_{\max}) \in \mathcal{P}_t$, is satisfied when output $y \mapsto f(x_{\min}, y)$ is maximal for $y = y_{\min}$ and $y \mapsto f(x_{\max}, y)$ is maximal for $y = y_{\max}$.

Lemma 5. *Suppose that the preceding assumption is in place. Then*

$$p_t^e \geq \hat{p}_t - \frac{L^\xi}{\underline{N}}.$$

As an immediate consequence, the additional assumption ensures that $\hat{p}_t$ gives a threshold probability up to which upper counter matching sets do satisfy the PAM-defining probabilities.

Theorem 4 (non-stationary Shimer and Smith with pair-specific production shocks)**.** *Suppose the preceding assumption is in place. Then for all $p \leq \hat{p}_t - \frac{L^\xi}{\underline{N}}$, $U_t(p)$ and $U_t(x;p)$ satisfy the PAM-defining properties.*

This result encompasses the original result due to Shimer and Smith (2000). In the limit case when ξ is negligible and the economy is stationary, $\hat{p}_t \to 1$, for otherwise some agent type would never match, leading to a zero option value of search. As they consider a continuum of types, corresponding to $\underline{N} \to \infty$, PAM can be recovered. More generally, our result establishes that with negligible pair-specific production shocks matching patterns exhibit PAM even in non-stationary environments (provided that search frictions are sufficiently large, so that no agent type can afford to reject all other agent types). In contrast, for diffuse pair-specific production shocks ξ, a large $\hat{p}_t$ will be difficult to satisfy. Thus we do not view this result as a vindication of PAM, but rather as one exploring its limits.

If search frictions upset PAM, why does PAM obtain in Shimer and Smith (2000)? The key difference is that in their model, $m_t(x, y)$ is not a representation of preferences over meetings. Consequently, our characterization of PAM in terms of preferences does not apply to their environment. However, this does not invalidate the robust prediction of preferences for NAM, i.e., $\overline{\mathcal{E}}_t \cup \underline{\mathcal{E}}_t$ is non-empty. In other words, the preference for NAM is not an artefact of pair-specific shocks. It is already invisibly present in Shimer and Smith's model.

To clarify this point, it is instructive to draw on the deterministic match surplus. In both frameworks, for all pairs in the NAM region $\overline{\mathcal{E}}_t$ and $\underline{\mathcal{E}}_t$, the following property obtains: for all $x_2 > x_1$ and $y_2 > y_1$, and $k \in \mathbb{R}$,

$$S_t(x_1, y_1) \geq k \text{ and } S_t(x_2, y_2) \geq k \implies S_t(x_1, y_2) \geq k \text{ and } S_t(x_2, y_1) \geq k$$

In words, take four types $x_1 < x_2$ and $y_1 < y_2$, such that both assortative pairs, (x_1, y_1) and (x_2, y_2), generate a deterministic surplus greater than a constant k. Then both non-assortative pairs, (x_1, y_2) and (x_2, y_1), generate a deterministic surplus at least greater than k. What differs in both framework is how $S_t(x, y)$ translates into matching probabilities. In our model, due to pair-specific shocks, the probability of matching is a monotone transformation of the deterministic surplus. Thereby, we get,

$$m_t(x_1, y_1) \geq p \text{ and } m_t(x_2, y_2) \geq p \overset{NAM}{\Longrightarrow} m_t(x_1, y_2) \geq p \text{ and } m_t(x_2, y_1) \geq p,$$

which is precisely the theoretical definition of NAM. In Shimer and Smith (2000), all pairs in this subset match with certainty because the deterministic surplus is positive is this region. They need the aforementioned boundary conditions to guarantee that the surplus is indeed positive within $\overline{\mathcal{E}}_t$ and $\underline{\mathcal{E}}_t$. Therefore, matching probabilities are identical across pairs, so PAM, but also NAM obtains:

$$m_t(x_1, y_1) = 1 \text{ and } m_t(x_2, y_2) = 1 \underset{PAM}{\overset{NAM}{\Longleftrightarrow}} m_t(x_1, y_2) = 1 \text{ and } m_t(x_2, y_1) = 1,$$

As a result, predictions differ simply because in Shimer and Smith's model the probability of matching imperfectly captures the heterogeneity in surplus across pairs.

6 Conclusion

We have presented here two forays into the theory of random search and matching. First, we provided an existence and uniqueness result of equilibrium. This casts doubt on the robustness of multiple self-fulfilling (non-stationary) equilibrium paths frequently reported in the literature. Secondly, we formulated a new insight into assortative matching under Nash bargaining: search frictions impede assortative matching from occurring because frictions disproportionately erode the value of search and hence the bargaining power of more productive agents. In consequence, unproductive agents prioritize matching with productive agents. As a main result, we provided a comparative characterization of matching patterns that holds away from the steady state and regardless of the degree of search frictions.

Our objective was to provide a theory simple enough to allow for theoretical insights into sorting, yet rich enough to be plausibly at the origin of empirically observed matching patterns. In line with our objective, we proposed a theory in which match outcomes were probabilistic and the economy non-stationary. We hope that this theory will further invigorate the ongoing quest of identifying sorting in the labor market and consider with greater attention the role of pair-specific production shocks and non-stationary dynamics. We equally hope that our sorting result provides a useful lens through which empirical researchers can view matching patterns.

Our theoretical inquiry indicates several lines for future research. First, we conjecture that our definition of probabilistic assortative matching obtains under reasonable conditions over the match output in the frictionless model. If that is proven to be true, it could help us delineate more clearly the trade-offs involved when adopting the frictionless matching framework for empirical research. Secondly, our main sorting results makes a number of empirical predictions. As is well known, unobserved heterogeneity of workers' types renders the refutation of those predictions a difficult task. We view it as imperative however that the attempt be made to verify whether the ordinal properties asserted are empirically valid. Finally, we believe that our framework is a suitable benchmark for applied theory; it could be studied more extensively to investigate labor market interventions such as the minimum wage. Here sorting, in particular the reallocation of workers from less to more productive firms as well as the length of employment spells has an important role to play. Relatedly, our uniqueness result is a step forward in that direction as it potentially allows to study welfare.

Appendix

A Introductory proofs

The probability of meeting k agents with types $\{y_1, ..., y_k\} \subset Y$ during time interval $(t_0, t_1]$ is given by

$$\frac{1}{k!}\prod_{\ell=1}^{k}\int_{t_0}^{t_1}\lambda_t(y_\ell)dt\ \exp\Big\{-\sum_{y\in Y}\int_{t_0}^{t_1}\lambda_t(y)dt\Big\},$$

and follows from the definition of the inhomogenous Poisson point process.

A.1 Remark 1

Proof. Internal consistence of the model requires that, for all $x \in X$ and $y \in Y$,

$$\lambda_t(y)\mu_t(x) = \lambda_t(x)\mu_t(y),$$

where either side expresses the flow number of meetings between agent types x and y. Fix $x \in X$ for which $\mu_t(x) > 0$. Then for all $y \in Y$

$$\lambda_t(y) = \beta_t^Y \mu_t(y) \quad \text{where} \quad \beta_t^Y = \frac{\lambda_t(x)}{\mu_t(x)}.$$

Following identical arguments there exists β_t^X such that $\lambda_t(x) = \beta_t^X \mu_t(x)$. Substituting back into the balance condition where $\mu_t(x), \mu_t(y)$ are non-zero we note that $\beta_t \equiv \beta_t^X = \beta_t^Y$. □

The proof implicitly assumes that there always exist x and y such that $\mu_t(x), \mu_t(y) > 0$. This will be a natural property of the model whenever the initial population of said types is non-zero.

As stated in the main text, the order of lemmata 1 and 2 is reversed for expositional purposes. We begin with lemma 2, which we use to establish lemma 1.

A.2 Proof of Lemma 2

Proof. We seek to transform the value of search into a martingale: according to Bayes' law for $\tau > t$, $P_0(\tau)(y|x) = P_0(t)(y|x) + \mathbb{P}(\text{don't match during } [0,t))P_t(\tau)(y|x)$, and therefore

$$P_t(\tau)(y|x) = \frac{P_0(\tau)(y|x) - P_0(t)(y|x)}{1 - \sum_{k=1}^{N} P_0(t)(k|x)}.$$

Then define

$$\begin{aligned}\tilde{V}_t(x) &\equiv e^{-\rho t}\big(1-\sum_{k=1}^{N}P_0(t)(k|x)\big)V_t(x)\\ &= \mathbb{E}\big[\int_t^T \sum_{y\in Y} e^{-\tau\rho}\pi_\tau(x,y)P_0(d\tau)(y|x) + e^{-\rho T}\big(1-\sum_{k=1}^{N}P_0(T)(k|x)\big)h(\mu_T)(x)\big|\mathcal{F}_t\big].\end{aligned}$$

The desired martingale is then given by

$$\begin{aligned}M_t(x) &\equiv \tilde{V}_t(x) + \int_0^t \sum_{y\in Y} e^{-\rho\tau}\pi_\tau(x,y)P_0(d\tau)(y|x)\\ &= \mathbb{E}\big[\int_0^T \sum_{y\in Y} e^{-\rho\tau}\pi_\tau(x,y)P_0(d\tau)(y|x) + e^{-\rho T}\big(1-\sum_{k=1}^{N}P_0(T)(k|x)\big)h(\mu_T)(x)\big|\mathcal{F}_t\big].\end{aligned}$$

And thus by the martingale representation theorem there exists a unique square integrable N-valued process $Z(x)$ such that

$$M_t(x) = M_T(x) - \int_t^T Z_\tau(x)\cdot dB_\tau.$$

Substituting back into the definition of $\tilde{V}_t(x)$, this gives

$$\begin{aligned}\tilde{V}_t(x) &= M_t(x) - \int_0^t \sum_{y\in Y} e^{-\rho\tau}\pi_\tau(x,y)P_0(d\tau)(y|x)\\ &= M_T(x) - \int_t^T Z_\tau(x)\cdot dB_\tau - \int_0^t \sum_{y\in Y} e^{-\rho\tau}\pi_\tau(x,y)P_0(d\tau)(y|x)\\ &= \tilde{V}_T(x) + \int_t^T \sum_{y\in Y} e^{-\rho\tau}\pi_\tau(x,y)P_0(d\tau)(y|x) - \int_t^T Z_\tau(x)\cdot dB_\tau\\ &= e^{-\rho T}\big(1-\sum_{k=1}^{N}P_0(T)(k|x)\big)h(\mu_T)(x) + \int_t^T \sum_{y\in Y} e^{-\rho\tau}\pi_\tau(x,y)P_0(d\tau)(y|x) - \int_t^T Z_\tau(x)\cdot dB_\tau,\end{aligned}$$

or, written in differential form,

$$d\tilde{V}_t(x) = -\sum_{y\in Y} e^{-\rho t}\pi_t(x,y)P_0(dt)(y|x) + Z_t(x)\cdot dB_t.$$

Now observe that due to Ito's lemma,

$$\begin{aligned}&d\tilde{V}_t(x) = d\big[e^{-\rho t}\big(1-\sum_{k=1}^{N}P_0(t)(k|x)\big)V_t(x)\big]\\ &= e^{-\rho t}\Big\{-\rho\big(1-\sum_{k=1}^{N}P_0(t)(k|x)\big)V_t(x)dt - V_t(x)\sum_{y\in Y}P_0(dt)(y|x) + \big(1-\sum_{k=1}^{N}P_0(t)(k|x)\big)dV_t(x)\Big\}.\end{aligned}$$

By identifying the two we obtain

$$
\begin{aligned}
&-\sum_{y\in Y} e^{-\rho t}\pi_t(x,y)P_0(dt)(y|x) + Z_t(x)\cdot dB_t \\
&= e^{-\rho t}\Big\{-\rho\big(1-\sum_{k=1}^{N}P_0(t)(k|x)\big)V_t(x)dt - V_t(x)\sum_{y\in Y}P_0(dt)(y|x) + \big(1-\sum_{k=1}^{N}P_0(t)(k|x)\big)dV_t(x)\Big\}.
\end{aligned}
$$

Or, equivalently,

$$
-\sum_{y\in Y}\big[\pi_t(x,y)-V_t(x)\big]\frac{P_0(dt)(y|x)}{1-\sum\limits_{k=1}^{N}P_0(t)(k|x)} + \rho V_t(x)dt + \tilde{Z}_t(x)\cdot dB_t = dV_t(x),
$$

where

$$
\tilde{Z}_t(x) = e^{\rho t}\frac{Z_t(x)}{1-\sum_{k=1}^{N}P_0(t)(k|x)}.
$$

Then note that $P_0(d\tau)(y|x)/\big(1-\sum\limits_{k=1}^{N}P_0(\tau)(k|x)\big) = P_t(d\tau)(y|x)$. Since the probability of meeting more than one agent in a given time interval of length ℓ is $o(\ell)$, $P_t(d\tau)(y|x)$ admits a neat representation: it is the hazard rate $\lambda_\tau(k|x)m_\tau(x,y)$ at which agent type x exits the search pool with an agent type y.

Substituting back into the former expression gives the BSDE

$$
V_t(x) = h(\mu_T)(x) + \int_t^T \big[\sum_{y\in Y}\big[\pi_\tau(x,y)-V_\tau(x)\big]\lambda_\tau(y|x)m_\tau(x,y) - \rho V_\tau(x)\big]d\tau - \int_t^T \tilde{Z}_t(x)\cdot dB_t,
$$

with endpoint constraint $V_T(x) = h(\mu_T)(x)$, as desired. □

Since $V_t(x) = \mathbb{E}\big[V_t(x)|\mathcal{F}_t\big]$, taking expectations we deduce the following corollary of lemma 2:

Corollary 4.

$$
V_t(x) = \mathbb{E}\Big[h_T + \int_t^T \big[\sum_{y\in Y}\big[\pi_\tau(x,y)-V_\tau(x)\big]\lambda_\tau(y|x)m_\tau(x,y) - \rho V_\tau(x)\big]d\tau\Big|\mathcal{F}_t\Big].
$$

The corollary is an immediate implication of the lemma, since the expectation of stochastic integrals of square integrable processes is zero.

A.3 Proof of Lemma 1

Proof. Consider (for the purposes of this proof) expected ex-post surplus conditional on matching, $\hat{S}^{\theta}_\tau(x,y) \equiv \pi^{\theta}_\tau(x,y) - V_\tau(x)$. According to the preceding corollary,

$$
\begin{aligned}
&W^{\theta^*}_t(x) - W^{\theta}_t(x) \\
&= \mathbb{E}\Big[\int_t^T \Big\{\sum_{y\in Y}\big[\hat{S}^{\theta^*}_\tau(x,y)m^{\theta^*}_\tau(x,y) - \hat{S}^{\theta}_\tau(x,y)m^{\theta}_\tau(x,y)\big]\lambda_\tau(y|x) - \rho\big[V_\tau(x) - W^{\theta}_\tau(x)\big]\Big\}d\tau\Big|\mathcal{F}_t\Big].
\end{aligned}
$$

Note that by virtue of the choice of θ^*:

$$\hat{S}^{\theta^*}_\tau(x,y)m^{\theta^*}_\tau(x,y) - \hat{S}^{\theta}_\tau(x,y)m^{\theta}_\tau(x,y) = \int\limits_{\theta^*_\tau(x,y)}^{\theta_\tau(x,y)} \frac{f(x,y)+\xi - V_\tau(x) - V_\tau(y)}{2}\Xi_\tau(x,y)(d\xi) \geq 0.$$

To simplify the notation, define for all $\tau \geq t$

$$u(\tau) \equiv -\mathbb{E}\Big[V_\tau(x) - W^\theta_\tau(x)\Big|\mathcal{F}_t\Big]$$
$$\alpha(\tau) \equiv -\mathbb{E}\Big[\int\limits_\tau^T \sum_{y=1}^N \big[S^{\theta^*}_r(x,y)m^{\theta^*}_r(x,y) - S^{\theta}_r(x,y)m^{\theta}_r(x,y)\big]\lambda_r(y|x)dr\Big|\mathcal{F}_t\Big].$$

Then

$$u(t) = \rho\int\limits_T^t u(\tau)d\tau + \alpha(t).$$

It remains to prove that $u(t) \leq 0$. This follows from Gröwall's lemma, since $\tau \mapsto \alpha(\tau)$ is non-positive and non-decreasing. Therefore

$$u(t) \leq \alpha(t)\exp\big\{-\rho(T-t)\big\} \leq 0.$$

□

B Uniqueness and existence: proof of theorem 1

B.1 Delarue's theorem

We present here the system of FBSDEs studied by Delarue (2002), where we slightly adapt the problem to our purposes. Our choice of notation follows Delarue (2002). We then proceed and restate appropriate assumptions under which he proves the well-posedness of the system, namely existence and uniqueness of a solution.[24]

Throughout, denote $||\cdot||$ the 1-norm, i.e., for any vector $v \in \mathbb{R}^N$ let $||v|| = \frac{1}{N}\sum\limits_{k=1}^N |v^k|$, where $|v^k|$ denotes the absolute value of the kth coordinate of v. Let

$$\tilde{f}:[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\times\to\mathbb{R}^N,$$
$$\tilde{g}:[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\times\to\mathbb{R}^N,$$
$$\tilde{\sigma}:[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N,$$
$$\tilde{h}:\mathbb{R}^N\to\mathbb{R}^N$$

measurable functions with respect to the Borelian σ-algebras.

[24] Delarue (2002) considers a slightly richer problem where $\tilde{X}_0$ is random, square integrable and measurable with respect to some σ-algebra $\mathcal{G}_0 : \mathcal{F}_0 \subseteq \mathcal{G}_0$; the dimensions of the forward- and backward SDE need not match, coefficients of $\tilde{f}$ and $\tilde{g}$ may in addition depend on $\tilde{Z}$. This more general description of FBSDEs does not add any economic insight to our problem.

For any $\mathbb{R}^N$-valued initial condition $\tilde{X}_0$ we are seeking $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{N\times N}$-valued and $\{\mathcal{F}_t\}$-progressively measurable processes $(\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t)_{t\in[0,T]}$, solution of the of FBSDEs:

$$\begin{cases} \forall t \in [0,T], \\ \tilde{X}_t = \tilde{X}_0 + \int\limits_0^t \tilde{f}(s, \tilde{X}_s, \tilde{Y}_s)ds + \int\limits_0^t \tilde{\sigma}(s, \tilde{X}_s, \tilde{Y}_s)dB_s \\ \tilde{Y}_t = \tilde{h}(X_T) + \int\limits_t^T \tilde{g}(s, \tilde{X}_s, \tilde{Y}_s)ds - \int\limits_t^T \tilde{Z}_s dB_s \\ \mathbb{E}\big[\int\limits_0^T \big((\tilde{X}_t)^2 + (\tilde{Y}_t)^2 + (\tilde{Z}_t)^2\big)dt\big] < \infty \end{cases} \tag{E'}$$

We now present the (slightly adapted) regularity conditions from Delarue (2002). The first condition is Lipschitz continuity of $\tilde{f}, \tilde{g}, \tilde{h}$ and $\tilde{\sigma}$ and corresponds to Delarue (2002)'s conditions (A1.1), (A1.2) and (A2.1). The second condition imposes a linear bound on those functions and corresponds to (A1.3) and (A2.2). The third condition imposes continuity of the drift and corresponds to (A1.4). Finally, the fourth condition imposes non-degeneracy of the variance of the diffusion process as well as continuity and corresponds to (A2.3) (non-degeneracy, after which Delarue (2002)'s article is named) and (A2.4).

Assumption A (Delarue (2002): assumptions (A1) and (A2))**.** *Functions $\tilde{f}, \tilde{g}, \tilde{h}$ and $\tilde{\sigma}$ satisfy assumption A if there exist two non-negative constants $\tilde{L}$ and $\tilde{\Lambda}$, as well as $\underline{\sigma} > 0$ such that:*

- $\forall t \in [0,T]$ *and* $(x,y), (x',y') \in \mathbb{R}^N \times \mathbb{R}^N$

$$\begin{aligned} ||\tilde{f}(t,x,y) - \tilde{f}(t,x,y')|| &\leq \tilde{L}||y-y'|| \\ ||\tilde{g}(t,x,y) - \tilde{g}(t,x',y)|| &\leq \tilde{L}||x-x'|| \\ ||\tilde{h}(x) - \tilde{h}(x')|| &\leq \tilde{L}||x-x'|| \\ ||\tilde{\sigma}(t,x,y) - \tilde{\sigma}(t,x',y')|| &\leq \tilde{L}^2(||x-x'||^2 + ||y-y'||^2) \end{aligned}$$

- $\forall t \in [0,T]$ *and* $(x,y), (x',y') \in \mathbb{R}^N \times \mathbb{R}^N$

$$\begin{aligned} ||\tilde{f}(t,x,y)|| &\leq \tilde{\Lambda}(1 + ||y||) \\ ||\tilde{g}(t,x,y)|| &\leq \tilde{\Lambda}(1 + ||y||) \\ ||\tilde{\sigma}(t,x,y)|| &\leq \tilde{\Lambda}(1 + ||y||) \\ ||\tilde{h}(x)|| &\leq \tilde{\Lambda} \end{aligned}$$

- $\forall t \in [0,T]$ *and* $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ *the functions* $u \mapsto \tilde{f}(t,u,y)$ *and* $v \mapsto \tilde{g}(t,x,v)$ *are continuous.*
- $\forall t \in [0,T]$ *and* $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ *and* $k \in \{1, ..., N\}$

$$\tilde{\sigma}^k(t,x,y) \geq \underline{\sigma},$$

  that is, the function $\tilde{\sigma}$ is uniformly bounded from below. Finally, $(t,x,y) \mapsto \tilde{\sigma}(t,x,y)$ is continuous.

We now proceed to state the first part of his main result:

Theorem A (Delarue (2002): theorem 2.6.1)**.** *Problem (E') admits a unique solution under assumption A.*

At the heart of this result is the connection of FBSDEs with a quasi-linear parabolic system of PDEs. This had first been presented by Ma, Protter and Yong in their four step scheme. Delarue (2002) improves on this initial result in a number of ways, notably by relaxing the smoothness of parameters as seen in assumption A.

Drawing on Delarue's theorem A, we may then formulate the proof of theorem 1 which establishes the existence of a unique equilibrium under assumption 1.

B.2 Transfer of notation

Rewrite the system of FBSDEs (E) as

$$\begin{aligned}\gamma_t(x) &= \int_0^t \Big[-\sum_{y\in Y} \beta_\tau e^{\gamma_\tau(y)} m_\tau(x,y) + \eta_\tau(x) - \frac{\big(\sigma_\tau(x)\big)^2}{2}\Big] d\tau + \int_0^t \sigma_\tau(x) dB_\tau \\ V_t(x) &= h_T + \int_t^T \alpha^X_\tau \sum_{y\in Y} \beta_\tau e^{\gamma_\tau(y)} m_\tau(x,y)\big[f(x,y) + \xi^e_\tau(x,y) - V_\tau(x) - V_\tau(y)\big] - \rho V_\tau(x)\big] d\tau \\ &\quad - \int_t^T Z_\tau(x)\cdot dB_\tau,\end{aligned}$$

where

$$\begin{aligned}\xi^e_t(x,y) &\equiv \frac{1}{m_t(x,y)} \int_{V_t(x)+V_t(y)-f(x,y)}^{\infty} \xi \Xi_t(d\xi) \\ m_t(x,y) &\equiv 1 - \Xi_t\big[V_t(x) + V_t(x) - f(x,y)\big]\end{aligned}$$

is the time t expected pair-specific production shock and meeting-contingent matching rate respectively between types x and y.

We then map our notation into Delarue's. Recall the type space is $X \cup Y = \{1, ..., N^X, N^X + 1, ..., N^X + N^Y \equiv N\}$. For any type $k \in X \cup Y$ denote $\chi(k)$ her own population: $\chi(k) = X$ if $k \in X$ and $\chi(k) = Y$ if $k \in Y$. Similarly, let $\chi^c(k) = X \cup Y \setminus \chi(k)$ the population of k's potential partners.

Denote $\tilde{X}^k_t = \gamma_t(k)$ and $\tilde{Y}^k_t = V_t(k)$ for any $k \in \{1, ..., N\}$. Finally, to abbreviate notation, denote $\Theta_t = (t, \tilde{X}_t, \tilde{Y}_t)$. Then identify

$$\begin{aligned}\tilde{f}^k(\Theta_t) &\equiv -\sum_{\ell\in\chi(k)} \beta(\Theta_t) \exp \tilde{X}^\ell_t \big[1 - \Xi(\Theta_t)(\tilde{Y}^k_t + \tilde{Y}^\ell_t - f(k,\ell))\big] + \eta(\Theta_t)(k) \\ \tilde{g}^k(\Theta_t) &\equiv \alpha^{\chi(k)}(\Theta_t) \sum_{\ell\in\chi^c(k)} \beta(\Theta_t) \exp \tilde{X}^\ell_t \big[1 - \Xi(\Theta_t)(\tilde{Y}^k_t + \tilde{Y}^\ell_t - f(k,\ell))\big] \\ &\qquad \cdot \big[f(k,\ell) + \xi^e_t(k,\ell) - \tilde{Y}^k_t - \tilde{Y}^\ell_t\big] - \rho \tilde{Y}^k_t \\ \tilde{\sigma}^k(\Theta_t) &\equiv \sigma(\Theta_t)(k) \\ \tilde{h}^k(\tilde{X}_T) &\equiv h(\tilde{X}_T).\end{aligned} \tag{T}$$

Under thus introduced notation the system E corresponds, as desired, to the system (E') studied by Delarue (2002).

B.3 Regularity conditions and proof

Lemma 6. *Suppose that the regularity condition 1 is in place. Then* $\tilde{f}, \tilde{g}, \tilde{\sigma}, \tilde{h}$ *as defined in* (T) *satisfy assumption A.*

Proof. We proceed in the same order as assumption A.

- Lipschitz continuity

Lipschitz continuity of $\tilde{h}$ and $\tilde{\sigma}$ in (γ_t, V_t) is required by assumption 1. As to $\tilde{f}$, denote $\Theta = (t, \tilde{X}, \tilde{Y})$ and $\Theta'_t = (t, \tilde{X}_t, \tilde{Y'}_t)$. Then

$$
\begin{aligned}
&\big|\tilde{f}^k(\Theta_t) - \tilde{f}^k(\Theta'_t)\big| \\
&\quad = \big|\sum_{\ell \in \chi(k)} \beta(\Theta_t) \exp \tilde{X}^\ell_t \big\{\Xi(\Theta_t)\big(\tilde{Y}^k_t + \tilde{Y}^\ell_t - f(k,\ell)\big) - \Xi(\Theta'_t)\big(\tilde{Y'}^k_t + \tilde{Y'}^\ell_t - f(k,\ell)\big)\big\} + \eta(\Theta_t) - \eta(\Theta'_t)\big| \\
&\quad \leq K^\beta \big\{ \sum_{\ell \in \chi(k)} \big|\Xi(\Theta_t)\big(\tilde{Y}^k_t + \tilde{Y}^\ell_t - f(k,\ell)\big) - \Xi(\Theta_t)\big(\tilde{Y'}^k_t + \tilde{Y'}^\ell_t - f(k,\ell)\big)\big| \\
&\qquad\quad + \sum_{\ell \in \chi(k)} \big|\Xi(\Theta_t)\big(\tilde{Y'}^k_t + \tilde{Y'}^\ell_t - f(k,\ell)\big) - \Xi(\Theta'_t)\big(\tilde{Y'}^k_t + \tilde{Y'}^\ell_t - f(k,\ell)\big)\big|\big\} + \big|\eta(\Theta_t) - \eta(\Theta'_t)\big| \\
&\quad \leq K^\beta \sum_{\ell \in \chi(k)} L^\xi \big|\tilde{Y}^k_t - \tilde{Y'}^k_t + \tilde{Y}^l_t - \tilde{Y'}^l_t\big| + K^\beta N L^\xi \big|\big|\tilde{Y}_t - \tilde{Y'}_t\big|\big| + L^\eta \big|\big|\tilde{Y}_t - \tilde{Y'}_t\big|\big| \\
&\quad \leq K^\beta \big(\#\chi(k)\, L^\xi \big|\tilde{Y}^k_t - \tilde{Y'}^k_t\big| + \sum_{\ell \in \chi(k)} L^\xi \big|\tilde{Y}^\ell_t - \tilde{Y'}^\ell_t\big|\big) + \big(K^\beta N L^\xi + L^\eta\big)\big|\big|\tilde{Y}_t - \tilde{Y'}_t\big|\big| \\
&\quad \leq K^\beta \big(N L^\xi \sum_{\ell=1}^N \big|\tilde{Y}^\ell_t - \tilde{Y'}^\ell_t\big| + \sum_{\ell=1}^N L^\xi \big|\tilde{Y}^\ell_t - \tilde{Y'}^\ell_t\big|\big) + \big(K^\beta N L^\xi + L^\eta\big)\big|\big|\tilde{Y}_t - \tilde{Y'}_t\big|\big| \\
&\quad \leq \big(K^\beta (2N+1) L^\xi + L^\eta\big)\big|\big|\tilde{Y}_t - \tilde{Y'}_t\big|\big|.
\end{aligned}
$$

Therefore $\big|\big|\tilde{f}(\Theta_t) - \tilde{f}(\Theta'_t)\big|\big| \leq N\big(K^\beta(2N+1)L^\xi + L^\eta\big)\big|\big|\tilde{Y}_t - \tilde{Y'}_t\big|\big|$ as desired.

As to $\tilde{g}$, denote $\Theta_t = (t, \tilde{X}_t, \tilde{Y}_t)$ and $\Theta'_t = (t, \tilde{X'}_t, \tilde{Y}_t)$. Likewise, denote $\xi^e_t(k,\ell) \equiv \xi^e(\Theta_t)(k,\ell)$ (the expected pair-specific production shock between types k and ℓ conditional on matching when the distribution function is $\Xi(\Theta_t)$ and the value of search is $\tilde{V}_t$). Then

$$
\begin{aligned}
&\tilde{g}^k(\Theta_t) - \tilde{g}^k(\Theta_t') \\
&\quad = \big(\alpha^{\chi(k)}(\Theta_t) - \alpha^{\chi(k)}(\Theta_t')\big) \sum_{\ell \in \chi^c(k)} \beta(\Theta_t) \exp \tilde{X}_t^\ell \big[1 - \Xi(\Theta_t)(\tilde{Y}_t^k + \tilde{Y}_t^\ell - f(k,\ell))\big] \\
&\qquad\qquad \big[f(k,\ell) + \xi_t^e(k,\ell) - \tilde{Y}_t^k - \tilde{Y}_t^\ell\big] \\
&\qquad + \alpha^{\chi(k)}(\Theta_t') \sum_{\ell \in \chi^c(k)} \big(\beta(\Theta_t) \exp \tilde{X}_t^\ell - \beta(\Theta_t') \exp \tilde{X'}_t^\ell\big) \big[1 - \Xi(\Theta_t)(\tilde{Y}_t^k + \tilde{Y}_t^\ell - f(k,\ell))\big] \\
&\qquad\qquad \big[f(k,\ell) + \xi_t^e(k,\ell) - \tilde{Y}_t^k - \tilde{Y}_t^\ell\big] \\
&\qquad + \alpha^{\chi(k)}(\Theta_t') \sum_{\ell \in \chi^c(k)} \beta(\Theta_t') \exp \tilde{X'}_t^\ell \\
&\qquad\qquad \Big\{ \big[\Xi(\Theta_t')(\tilde{Y}_t^k + \tilde{Y}_t^\ell - f(k,\ell)) - \Xi(\Theta_t)(\tilde{Y}_t^k + \tilde{Y}_t^\ell - f(k,\ell))\big] \big[f(k,\ell) - \tilde{Y}_t^k - \tilde{Y}_t^\ell\big] \\
&\qquad\qquad\quad + \int_{\tilde{Y}_t^k + \tilde{Y}_t^\ell - f(k,\ell)}^{\bar{\xi}} \xi \, \Xi(\Theta_t)(d\xi) - \int_{\tilde{Y}_t^k + \tilde{Y}_t^\ell - f(k,\ell)}^{\bar{\xi}} \xi \, \Xi(\Theta_t')(d\xi) \Big\}.
\end{aligned}
$$

Now observe that $\tilde{X}_t \mapsto \beta(t, \tilde{X}_t, \tilde{Y}_t) \exp \tilde{X}_t^k$ is continuous and bounded, whence Lipschitz continuous. Denote associated Lipschitz constant L^β. Further observe that, due to Lipschitz continuity of the density Ξ',

$$
\begin{aligned}
&\Big| \int_{\tilde{Y}_t^k + \tilde{Y}_t^\ell - f(k,\ell)}^{\bar{\xi}} \xi \, \Xi(\Theta_t)(d\xi) - \int_{\tilde{Y}_t^k + \tilde{Y}_t^\ell - f(k,\ell)}^{\bar{\xi}} \xi \, \Xi(\Theta_t')(d\xi) \Big| \\
&\qquad \le \int_{\tilde{Y}_t^k + \tilde{Y}_t^\ell - f(k,\ell)}^{\bar{\xi}} \xi \, \big|\Xi'(\Theta_t)(\xi) - \Xi'(\Theta_t')(\xi)\big| d\xi \le (\bar{\xi} - \underline{\xi}) L^\xi \big|\big|\tilde{X}_t - \tilde{X'}_t\big|\big|.
\end{aligned}
$$

Finally, for ease of notation, denote $\Pi = \max_{k,\ell} f(k,\ell) + \bar{\xi}$. It follows that

$$
\begin{aligned}
&\big|\tilde{g}^k(\Theta_t) - \tilde{g}^k(\Theta_t')\big| \\
&\quad \le L^\alpha \big|\big|\tilde{X}_t - \tilde{X'}_t\big|\big| N K^\beta \Pi \\
&\qquad + N L^\beta \big|\big|\tilde{X}_t - \tilde{X'}_t\big|\big| \Pi \\
&\qquad + N K^\beta L^\xi \big|\big|\tilde{X}_t - \tilde{X'}_t\big|\big| \Pi \\
&\qquad + N K^\beta (\bar{\xi} - \underline{\xi}) L^\xi \big|\big|\tilde{X}_t - \tilde{X'}_t\big|\big| \\
&\quad = N \Big(K^\beta \big(L^\alpha \Pi + L^\xi \Pi + (\bar{\xi} - \underline{\xi}) L^\xi\big) + L^\beta \Pi \Big) \big|\big|\tilde{X}_t - \tilde{X'}_t\big|\big|,
\end{aligned}
$$

so that

$$
\big|\big|\tilde{g}(\Theta_t) - \tilde{g}(\Theta_t')\big|\big| \le (N)^2 \Big(K^\beta \big(L^\alpha \Pi + L^\xi \Pi + (\bar{\xi} - \underline{\xi}) L^\xi\big) + L^\beta \Pi \Big) \big|\big|\tilde{X}_t - \tilde{X'}_t\big|\big|.
$$

- Boundedness

$\tilde{h}$ and $\tilde{\sigma}$ are bounded by assumption. Furthermore

$$\left|\tilde{f}^k(\Theta_t)\right| \leq NK^\beta + K^\eta \quad \text{and} \quad \left|\tilde{g}^k(\Theta_t)\right| \leq NK^\beta \Pi.$$

Multiplying by the number of types N gives the desired bounds of $\tilde{f}$ and $\tilde{g}$.

- Continuity

$\tilde{X}_t \mapsto \tilde{f}(t, \tilde{X}_t, \tilde{Y}_t)$, being the sum and product of continuous functions (in $\tilde{X}_t$) is continuous, and so is $\tilde{Y}_t \mapsto \tilde{g}(t, \tilde{X}_t, \tilde{Y}_t)$.

- Non-degeneracy

Non-degeneracy of σ follows from the last item of assumption A. □

The proof of theorem 1 follows immediately:

Proof. It suffices to prove that there exists a unique solution to the system of FBSDEs (E). Having verified in the preceding lemma that assumption A holds under stated regularity conditions 1, this follows from theorem A due to Delarue (2002). □

C Single-peaked preferences

C.1 Proof of Lemma 3

Proof. Fix arbitrary types x_1, x_2 in X (a symmetric proof applies for types in Y) and time t. Write $\bar{P}_t(\tau)(x) \equiv \sum_{y \in Y} P_t(\tau)(y|x)$. Let θ such that $m^\theta_\tau(x_2, y) = m_\tau(x_1, y)$ for all $\tau \in [0, T]$, types $y \in Y$ and across states $\omega \in \Omega$ (so that x_2 matches for identical realizations of ξ as x_1). Denote for $k = 1, 2, ...$

$$M^k_t(y|x_1) = \mathbb{E}\Big[\int_{\tau_0=t}^{T}\int_{\tau_1}^{T}\cdots\int_{\tau_{k-1}}^{T} e^{-\rho(\tau_k - t)}\alpha^X_{\tau_k} P_{\tau_{k-1}}(d\tau_k)(y|x_1) \prod_{l=k-1}^{1}(1-\alpha^X_{\tau_l})\bar{P}_{\tau_{l-1}}(d\tau_l)(x_1)\Big|\mathcal{F}_t\Big]$$

$$R^k_t(x_1, x_2) = \mathbb{E}\Big[\int_{\tau_0=t}^{T}\int_{\tau_1}^{T}\cdots\int_{\tau_{k-1}}^{T} \big(V_{\tau_k}(x_2) - V_{\tau_k}(x_1)\big)e^{-\rho(\tau_k - t)} \prod_{l=k}^{1}(1-\alpha^X_{\tau_l})\bar{P}_{\tau_{l-1}}(d\tau_l)(x_1)\Big|\mathcal{F}_t\Big].$$

We then prove by induction that

$$V_t(x_2) - V_t(x_1) \geq \sum_{y \in Y}\left[f(x_2, y) - f(x_1, y)\right]\sum_{l=1}^{k} M^l_t(y|x_1) + R^k_t(x_1, x_2).$$

Indeed, the optimality condition $V_\tau(x_2) \geq W^\theta_\tau(x_2)$ gives

$$\begin{aligned}
V_t(x_2) - V_t(x_1) &\geq W^\theta_t(x_2) - V_t(x_1) \\
&= \mathbb{E}\Big[\int_t^T \sum_{y\in Y} \big[\pi^\theta_\tau(x_2,y) - \pi_\tau(x_1,y)\big] e^{-\rho(\tau-t)} P_t(d\tau)(y|x_1)\Big|\mathcal{F}_t\Big] \\
&= \sum_{y\in Y} \big[f(x_2,y) - f(x_1,y)\big] \underbrace{\mathbb{E}\Big[\int_t^T e^{-\rho(\tau-t)} \alpha^X_\tau P_t(d\tau)(y|x_1)\Big|\mathcal{F}_t\Big]}_{M^1_t(y|x_1)} \\
&\qquad + \underbrace{\mathbb{E}\Big[\int_t^T \big(V_\tau(x_2) - V_\tau(x_1)\big) e^{-\rho(\tau-t)}(1-\alpha^X_\tau)\bar{P}_t(d\tau)(x_1)\Big|\mathcal{F}_t\Big]}_{R^1_t(x_1,x_2)}.
\end{aligned}$$

Now suppose by induction hypothesis that

$$V_t(x_2) - V_t(x_1) \geq \sum_{y\in Y} \big[f(x_2,y) - f(x_1,y)\big] \sum_{l=1}^{k-1} M^l_t(y|x_1) + R^{k-1}_t(x_1,x_2).$$

We show that

$$R^{k-1}_t(x_1,x_2) \geq \sum_{y\in Y} \big[f(x_2,y) - f(x_1,y)\big] M^k_t(y|x_1) + R^k_t(x_1,x_2)$$

from which the claim follows. Indeed, once more applying the optimality condition gives

$$\begin{aligned}
R^{k-1}_t(x_1,x_2) &\geq \mathbb{E}\Big[\int_{\tau_0=t}^T \int_{\tau_1}^T \cdots \int_{\tau_{k-2}}^T \big(W^\theta_{\tau_{k-1}}(x_2) - V_{\tau_{k-1}}(x_1)\big) e^{-\rho(\tau_{k-1}-t)} \prod_{l=k-1}^{1} (1-\alpha^X_{\tau_l})\bar{P}_{\tau_{l-1}}(d\tau_l)(x_1)\Big|\mathcal{F}_t\Big] \\
&= \mathbb{E}\Big[\int_{\tau_0=t}^T \int_{\tau_1}^T \cdots \int_{\tau_{k-2}}^T \Big[\int_{\tau_{k-1}}^T \sum_{y\in Y} \big[\pi^\theta_{\tau_k}(x_2,y) - \pi_{\tau_k}(x_1,y)\big] e^{-\rho(\tau_k-\tau_{k-1})} P_{\tau_{k-1}}(d\tau_k)(y|x_1)\Big] \\
&\qquad e^{-\rho(\tau_{k-1}-t)} \prod_{l=k-1}^{1} (1-\alpha^X_{\tau_l})\bar{P}_{\tau_{l-1}}(d\tau_l)(x_1)\Big|\mathcal{F}_t\Big] \\
&= \sum_{y\in Y} \big[f(x_2,y) - f(x_1,y)\big] \\
&\qquad \underbrace{\mathbb{E}\Big[\int_{\tau_0=t}^T \int_{\tau_1}^T \cdots \int_{\tau_{k-1}}^T e^{-\rho(\tau_k-t)} \alpha^X_{\tau_k} P_{\tau_{k-1}}(d\tau_k)(y|x_1) \prod_{l=k-1}^{1} (1-\alpha^X_{\tau_l})\bar{P}_{\tau_{l-1}}(d\tau_l)(x_1)\Big|\mathcal{F}_t\Big]}_{M^k_t(y|x_1)} \\
&\quad + \underbrace{\mathbb{E}\Big[\int_{\tau_0=t}^T \int_{\tau_1}^T \cdots \int_{\tau_{k-1}}^T \big(V_{\tau_k}(x_2) - V_{\tau_k}(x_1)\big) e^{-\rho(\tau_k-t)} \prod_{l=k}^{1}(1-\alpha^X_{\tau_l})\bar{P}_{\tau_{l-1}}(d\tau_l)(x_1)\Big|\mathcal{F}_t\Big]}_{R^k_t(x_1,x_2)}.
\end{aligned}$$

Then define

$$Q_t(y|x_1) = \lim_{k\to\infty} \sum_{l=1}^{k} M_t^l(y|x_1).$$

It remains to show that

$$\sum_{y\in Y} Q_t(y|x_1) \leq 1 \quad \text{and} \quad \lim_{k\to\infty} R_t^k(x_1, x_2) = 0 \quad \text{almost surely.}$$

We here prove this result when α_t^X is constant equal to α^X. A similarly straightforward result can be shown when the meeting rate appropriately bounded. The proof of the general result is deferred to a future version of this work. First note that

$$R_t^k(x_1, x_2) \leq (1 - \alpha^X)^k$$

Further observe that

$$\sum_{y\in Y} \int_t^T e^{-\rho(\tau - t)} \alpha^X P_t(d\tau)(y|x_1) \leq \alpha^X,$$

for the LHS describes the discounted expected probability times the bargaining weight. It follows that

$$\sum_{y\in Y} M_t^k(y|x_1) \leq \alpha^X (1 - \alpha_\tau^X)_1^{k-1},$$

whence

$$\sum_{y\in Y} Q_t(y|x_1) = \sum_{k\geq 1} \sum_{y\in Y} M_t^k(y|x_1) \leq \sum_{k\geq 1} \alpha^X (1 - \alpha^X)^{k-1} = 1.$$

□

C.2 Proof of theorem 2

We here prove convexity of matching sets. This proof relies on two intermediate observations. First, we need to establish quasi-concavity of output f. Clearly, in thin markets the value of search may be very small. Then for ex-ante surplus $S_t(x, y)$ to be quasi-concave, so must be output f:

Lemma 7. *If f is supermodular and log supermodular in differences with respect to y, then $f(x, y)$ is quasi-concave with respect to y for any $x > x_{min}$.*

Proof. Suppose that there exist $x_1 > x_0$ and $y_3 > y_2 > y_1$ such that $f(x_1, y_1) > f(x_1, y_2)$ and $f(x_1, y_3) > f(x_1, y_2)$. Supermodularity ensures that $f(x_1, y_3) - f(x_1, y_2) > f(x_0, y_3) - f(x_0, y_2)$. Multiplying on both side by $\frac{1}{f(x_1,y_2) - f(x_1,y_1)} < 0$, we obtain:

$$\frac{f(x_0, y_3) - f(x_0, y_2)}{f(x_1, y_2) - f(x_1, y_1)} > \frac{f(x_1, y_3) - f(x_1, y_2)}{f(x_1, y_2) - f(x_1, y_1)}.$$

We can construct an lower bound of the right hand side using log supermodularity in differences:

$$\frac{f(x_0,y_3)-f(x_0,y_2)}{f(x_1,y_2)-f(x_1,y_1)} > \frac{f(x_1,y_3)-f(x_1,y_2)}{f(x_1,y_2)-f(x_1,y_1)} \geq \frac{f(x_0,y_3)-f(x_0,y_2)}{f(x_0,y_2)-f(x_0,y_1)}.$$

Case 1: if $f(x_0,y_3)-f(x_0,y_2) \geq 0$ then the above inequalities imply that:

$$\frac{1}{f(x_1,y_2)-f(x_1,y_1)} > \frac{1}{f(x_0,y_2)-f(x_0,y_1)}.$$

If $f(x_0,y_3)-f(x_0,y_2) \geq 0$ then log supermodularity in differences imply that $f(x_0,y_2)-f(x_0,y_1) \leq 0$, and so the above inequality contradicts supermodularity: $f(x_1,y_2)-f(x_1,y_1) > f(x_0,y_2)-f(x_0,y_1)$.

Case 2: if $f(x_0,y_3)-f(x_0,y_2) \leq 0$ then,

$$\frac{1}{f(x_1,y_2)-f(x_1,y_1)} < \frac{1}{f(x_0,y_2)-f(x_0,y_1)}.$$

If $f(x_0,y_3)-f(x_0,y_2) \leq 0$ then log supermodularity in differences imply that $f(x_0,y_2)-f(x_0,y_1) \geq 0$ and so the above inequality contradicts supermodularity: $f(x_1,y_2)-f(x_1,y_1) > f(x_0,y_2)-f(x_0,y_1)$.

□

Secondly, log supermodularity in differences plays an important role in the proof, because it relates to preferences over risk-preferences. More specifically, Pratt (1964) shows that *the following statements are equivalent for any two agent types $x_1 < x_2$ with respective utility functions $y \mapsto h(x_i,y)$, $x_i \in \{x_1,x_2\}$:*

1. *$h(x,y)$ is log supermodular in differences with respect to y;*
2. *Agent type x_1 is more risk-averse than agent type x_2; that is, x_1 does not accept a lottery that is rejected by x_2.*

We then proceed to the proof of the convexity result:

Proof. We provide a proof by contraposition. Suppose there exists a type $x_1 \in X$ whose surplus function $y \mapsto S_t(x_1,y)$ at a given time t and state ω was not strictly quasi-concave. Then, there exist types $y_3 > y_2 > y_1$ such that

$$S_t^\omega(x_1,y_3) \geq S_t^\omega(x_1,y_2) \quad \text{and} \quad S_t^\omega(x_1,y_1) \geq S_t^\omega(x_1,y_2)$$

which is equivalent to

$$V_t^\omega(y_3)-V_t^\omega(y_2) \leq f(x_1,y_3)-f(x_1,y_2) \quad \text{and} \quad V_t^\omega(y_2)-V_t^\omega(y_1) \geq f(x_1,y_2)-f(x_1,y_1).$$

Due to lemma **??**, there exist ω-contingent non-negative weights $Q_t^\omega(\cdot|y_2): X \to \mathbb{R}_+$ with $Q_t^\omega(y_2) \equiv \sum_{x\in X} Q_t^\omega(x|y_2) \leq 1$ satisfying

$$\sum_{x\in X}[f(x,y_2)-f(x,y_1)]Q_t^\omega(x|y_2) \geq f(x_1,y_2)-f(x_1,y_1)$$
$$\sum_{x\in X}[f(x,y_3)-f(x,y_2)]Q_t^\omega(x|y_2) \leq f(x_1,y_3)-f(x_1,y_2).$$

Now distinguish between two cases. Suppose (i) that $f(x_1, y_2) - f(x_1, y_1) \geq 0$. Then let x_2 the largest type in $[0, 1]$ such that

$$\frac{1}{Q_t^\omega(y_2)}\big(f(x_1, y_2) - f(x_1, y_1)\big) \geq f(x_2, y_2) - f(x_2, y_1).$$

Due to supermodularity, the fact that $Q_t^\omega(y_2) \in (0, 1)$, and $f(x_1, y_2) - f(x_1, y_1) \geq 0$ we have that $x_2 \geq x_1$. Moreover, unless $x_2 = 1$, due to the intermediate value theorem the preceding holds as an equality. Now, if $x_2 = 1$, supermodularity alone implies that

$$\sum_{x \in X}[f(x, y_3) - f(x, y_2)]\frac{Q_t^\omega(x|y_2)}{Q_t^\omega(y_2)} \leq f(1, y_3) - f(1, y_2).$$

Otherwise, log supermodularity in differences of f with respect to y gives

$$\begin{aligned}
\frac{1}{Q_t^\omega(y_2)}(f(x_1, y_3) - f(x_1, y_2)) &= \frac{f(x_1, y_3) - f(x_1, y_2)}{f(x_2, y_3) - f(x_2, y_2)}\frac{1}{Q_t^\omega(y_2)}\big(f(x_2, y_3) - f(x_2, y_2)\big) \\
&\leq \frac{f(x_1, y_2) - f(x_1, y_1)}{f(x_2, y_2) - f(x_2, y_1)}\frac{1}{Q_t^\omega(y_2)}\big(f(x_2, y_3) - f(x_2, y_2)\big) = f(x_2, y_3) - f(x_2, y_2).
\end{aligned}$$

And so, there exists $\hat{x} = x_2$, such that

$$\begin{aligned}
&\sum_{x \in X}[f(x, y_2) - f(x, y_1)]\frac{Q_t^\omega(x|y_2)}{Q_t^\omega(y_2)} \geq f(\hat{x}, y_2) - f(\hat{x}, y_1) \\
&\sum_{x \in X}[f(x, y_3) - f(x, y_2)]\frac{Q_t^\omega(x|y_2)}{Q_t^\omega(y_2)} \leq f(\hat{x}, y_3) - f(\hat{x}, y_2).
\end{aligned}$$

Now suppose (ii) that $f(x_1, y_2) - f(x_1, y_1) < 0$. Since f is quasi-concave, also $f(x_1, y_3) - f(x_1, y_2) < 0$. We apply a symmetric reasoning as in (i): let x_0 the smallest type in $[0, 1]$ such that

$$\frac{1}{Q_t^\omega(y_2)}\big(f(x_1, y_3) - f(x_1, y_2)\big) \leq f(x_0, y_3) - f(x_0, y_2).$$

Due to supermodularity, the fact that $Q_t^\omega(y_2) \in (0, 1)$, and $f(x_1, y_3) - f(x_1, y_2) < 0$ we have that $x_0 \leq x_1$. Moreover, unless $x_0 = 0$, due to the intermediate value theorem the preceding holds as an equality. Now, if $x_0 = 0$, supermodularity alone implies that

$$\sum_{x \in X}[f(x, y_2) - f(x, y_1)]\frac{Q_t^\omega(x|y_2)}{Q_t^\omega(y_2)} \geq f(0, y_2) - f(0, y_1).$$

Otherwise, log supermodularity in differences of f with respect to y and the fact that $f(x_0, y_2) - f(x_0, y_1) < 0$ give

$$\begin{aligned}
\frac{1}{Q_t^\omega(y_2)}(f(x_1, y_2) - f(x_1, y_1)) &= \frac{f(x_1, y_2) - f(x_1, y_1)}{f(x_0, y_2) - f(x_0, y_1)}\frac{1}{Q_t^\omega(y_2)}\big(f(x_0, y_2) - f(x_0, y_1)\big) \\
&\geq \frac{f(x_1, y_3) - f(x_1, y_2)}{f(x_0, y_3) - f(x_0, y_2)}\frac{1}{Q_t^\omega(y_2)}\big(f(x_0, y_2) - f(x_0, y_1)\big) = f(x_0, y_2) - f(x_0, y_1).
\end{aligned}$$

And so, as in (i), there exists $\hat{x} = x_0$, such that

$$\sum_{x \in X} [f(x, y_2) - f(x, y_1)] \frac{Q_t^\omega(x|y_2)}{Q_t^\omega(y_2)} \geq f(\hat{x}, y_2) - f(\hat{x}, y_1)$$
$$\sum_{x \in X} [f(x, y_3) - f(x, y_2)] \frac{Q_t^\omega(x|y_2)}{Q_t^\omega(y_2)} \leq f(\hat{x}, y_3) - f(\hat{x}, y_2).$$

This pair of inequalities is of course equivalent to summation over differences:

$$\sum_{x \in X} \sum_{\substack{y \in Y \\ y_1 \leq y < y_2}} \triangle_y f(x, y) \frac{Q_t^\omega(x|y_2)}{Q_t^\omega(y_2)} \geq \sum_{\substack{y \in Y \\ y_1 \leq y < y_2}} \triangle_y f(\hat{x}, y)$$
$$\sum_{x \in X} \sum_{\substack{y \in Y \\ y_2 \leq y < y_3}} \triangle_y f(x, y) \frac{Q_t^\omega(x|y_2)}{Q_t^\omega(y_2)} \leq \sum_{\substack{y \in Y \\ y_2 \leq y < y_3}} \triangle_y f(\hat{x}, y).$$

In particular, this means there exist $y' : y_1 \leq y' < y_2$ and $y'' : y_2 \leq y'' < y_3$ such that

$$\sum_{x \in X} \triangle_y f(x, y') \frac{Q_t^\omega(x|y_2)}{Q_t^\omega(y_2)} \geq \triangle_y f(\hat{x}, y')$$
$$\sum_{x \in X} \triangle_y f(x, y'') \frac{Q_t^\omega(x|y_2)}{Q_t^\omega(y_2)} \leq \triangle_y f(\hat{x}, y'').$$

This runs counter to the characterization of log supermodularity in differences of $\triangle_y f$ in terms of risk preferences; interpret $x \mapsto \triangle_y f(\cdot, y)$ as agent type y's utility function. Then agent type y'' prefers the safe outcome $\hat{x}$ over a lottery, whereas agent type y', of lower rank than y'', prefers the lottery when facing the identical decision problem, in spite of Pratt's theorem. □

C.3 Observation: non single-peaked wages

We here make a remark that is interesting in its own right: provided that preferences are single-peaked, we show here that expected equilibrium payoffs, denoted $\pi_t^e(x, y)$, are not necessarily single-peaked. First, expected payoffs can be expressed as follows:

$$\pi_t^e(x, y) \equiv V_t(x) + \alpha_t^X S_t(x, y) + \frac{\alpha_t^X}{1 - \Xi_t(-S_t(x, y))} \int_{-S_t(x,y)}^{\bar{\xi}} \xi \, \Xi_t(d\xi).$$

We are interested in the shape of this function with respect to y. As $V_t(x)$ does not depend on y, it suffices to consider the second and third term. If those terms were a monotone transformation of S, π_t^e would inherit the properties of S and be single-peaked. The following reveals that absent further assumptions regarding Ξ this need not be the case. Consider

$$S \mapsto \alpha_t^X S + \frac{\alpha_t^X}{1 - \Xi_t(-S)} \int_{-S}^{\bar{\xi}} \xi \, \Xi_t(d\xi).$$

Taking the derivative with respect to S (using the Leibniz integral rule) we obtain

$$1 - \frac{\Xi_t'(-S)}{(1-\Xi_t(-S))^2} \int_{-S}^{\bar{\xi}} \xi \, \Xi_t(d\xi) + \frac{\Xi_t'(-S)S}{1-\Xi_t(-S)},$$

which boils down to

$$FOC: \quad 1 - \frac{\Xi_t'(-S)}{1-\Xi_t(-S)} \mathbb{E}^{\xi}\left[\xi - S \mid \xi + S\right].$$

Observe that this expression is not necessarily non-negative. It follows that, unlike meeting-contingent match probabilities and surplus, match payoffs conditional on matching need not be single-peaked.

D Assortative matching: proofs

D.1 Illustrative results

Non-emptiness + lattice $\Longrightarrow$ convexity

Proposition 4. *Suppose that $U_t(y;p)$ is non-empty for all $y \in Y$. Then if $U_t(p)$ is a lattice, then $U_t(x;p)$ is convex for all $x \in X$.*

The proof is an immediate adaptation from Shimer and Smith (2000).

Proof. Suppose that there exist x and $y_3 > y_2 > y_1$ such that y_3 and y_1 are in $U_t(x;p)$, but y_2 is not. Or, equivalently, $(x, y_1), (x, y_3) \in U_t(p)$, but $(x, y_2) \notin U_t(p)$. Since $U_t(y_2;p)$ is non empty, there exists a type $x' \in U_t(y_2;p)$. If $x' > x$, the fact that $(y_3, x), (y_2, x') \in U_t(p)$ and the lattice property imply that $(x, y_2) \in U_t(p)$. Using the same reasoning, we find a contradiction for the case $x' < x$. □

Proof of proposition 5

Proposition 5. *Suppose that $U_t(x;p)$ is convex for all agent types. Then $U_t(p)$ is a lattice if and only if $x \mapsto l_t(x;p)$ and $x \mapsto u_t(x;p)$ are non-decreasing.*

Proof. Maintain throughout that $U_t(x;p)$ is convex. ($\Longleftarrow$). Suppose that $x \mapsto l_t(x;p)$ as well as $x \mapsto u_t(x;p)$ are non-decreasing. Consider $x_1 < x_2$ and $y_1 < y_2$ such that (x_1, y_2) and (x_2, y_1) in $U_t(p)$. Then $(x_1, y_2) \in U_t(p)$ implies that $y_2 \le u(x_1;p)$. And due to $x \mapsto u_t(x;p)$ being non-decreasing, $y_2 \le u_t(x_2;p)$. Likewise, $(x_2, y_1) \in U_t(p)$ implies that $y_1 \ge l(x_2;p)$, and due to $x \mapsto l_t(x;p)$ being non-decreasing, $y_2 \ge l_t(x_2;p)$. As a result, $l_t(x_2;p) \le y_2 \le u_t(x_2;p)$. Since $U_t(x_2;p)$ is convex, this implies that y_2 in $U_t(x_2;p)$, or that $(x_2, y_2) \in U_t(p)$. Using an identical reasoning, it follows that $(x_1, y_1) \in U_t(p)$.

($\Longrightarrow$). Let $U_t(p)$ a lattice. Consider arbitrary $x_1 < x_2$. We show that $u_t(x_1;p) \le u_t(x_2;p)$. If not, for $(x_1, u_t(x_1,p)) \in U_t(p)$, $(x_2, u_t(x_2,p)) \in U_t(p)$, the lattice property implies that $(x_2, u_t(x_1,p)) \in U_t(p)$, or that $u_t(x_1,p) \in \{y : m_t(x_2,y) \ge p\}$. It follows that $u_t(x_2;p) \equiv \max\{y : m_t(x_2,y) \ge p\} \ge u_t(x_1,p)$ as was to be shown. An identical reasoning establishes the monotonicity of $l_t(x;p)$. □

D.2 Proof of the preference representation

We here provide three results that make a connection between preferences and PAM, and jointly establish an equivalent representation of PAM, namely proposition 2.

Proposition 6. *Suppose that preferences $\succsim_t^x$ and $\succsim_t^y$ satisfy weak single-crossing,and are weakly single-peaked and reciprocal. Then upper contour matching sets $U_t(p)$ are a lattice for all p.*

Proof. Pick any four types $x_1 < x_2$ and $y_1 < y_2$. We show that

$$\min\{m_t(x_1, y_1), m_t(x_2, y_2)\} \geq \min\{m_t(x_2, y_1), m_t(x_1, y_2)\},$$

thus proving the lattice property. Distinguish between two cases:
case (i) $m_t(x_1, y_2) > m_t(x_1, y_1)$. Weak single-crossing implies that $m_t(x_2, y_2) \geq m_t(x_2, y_1)$. Then for the preceding min-min condition to hold it suffices that $m_t(x_1, y_1) \geq m_t(x_2, y_1)$. Suppose by contradiction that this held with strictly reversed sign. We are thus left with two inequalities. The first inequality, $m_t(x_1, y_2) > m_t(x_1, y_1)$, taken together with weakly single-peaked preferences, implies that the set of agent type x_1 preferred partners' types strictly majorizes agent type y_1, therefore $y_1 < \underline{y}_t(x_1)$. The second inequality, $m_t(x_1, y_1) < m_t(x_2, y_1)$, taken together with weakly single-peaked preferences, implies that the set of agent type y_1 preferred partners' types strictly majorizes x_1, therefore $x_1 < \underline{x}_t(y_1)$. As both conditions cannot obtain jointly under reciprocity, this poses the desired contradiction in case (i).
Case (ii) $m_t(x_1, y_2) \leq m_t(x_1, y_1)$. Then for the preceding min-min condition to hold it suffices that $m_t(x_2, y_2) \geq \min\{m_t(x_1, y_2); m_t(x_2, y_1)\}$. Suppose by contradiction that this held with strictly reversed sign: then $m_t(x_2, y_2) < m_t(x_1, y_2)$ and $m_t(x_2, y_2) < m_t(x_2, y_1)$. The first inequality, taken together with weakly single-peaked preferences, implies that the set of agent type y_2 preferred partners' types is inferior to agent type x_2, therefore $x_2 > \overline{x}_t(y_1)$. The second inequality, taken together with weakly single-peaked preferences, implies that the set of agent type x_2 preferred partners' types is inferior to agent type y_2, therefore $y_2 > \overline{y}_t(x_2)$. As both conditions cannot obtain jointly under reciprocity, this poses the desired contradiction in case (ii). □

Proposition 7. *Suppose that upper contour matching sets $U_t(p)$ are a lattice for all p. Then the preference relation $\succsim_t^x$ satisfies weak single-crossing.*

Proof. Suppose by contradiction that $m_t(x, y)$ did not satisfy weak single-crossing, i.e., suppose there exist state ω, time t and four types $x_2 > x_1$ yet $y_2 > y_1$ such that $m_t^\omega(x_1, y_2) > m_t^\omega(x_1, y_1)$ and $m_t^\omega(x_2, y_2) < m_t^\omega(x_2, y_1)$. $U_t(p)$ being a lattice for all p requires that $\min\{m_t^\omega(x_1, y_1), m_t^\omega(x_2, y_2)\} \geq \min\{m_t^\omega(x_1, y_2), m_t^\omega(x_2, y_1)\}$. For convenience, denote $a \equiv m_t^\omega(x_1, y_1)$, $b \equiv m_t^\omega(x_2, y_2)$, $c \equiv m_t^\omega(x_1, y_2)$, $d \equiv m_t^\omega(x_2, y_1)$. Then the problem is to contradict the following set of inequalities:

$$c > a,\, d > b \quad \text{and} \quad \min\{a, b\} \geq \min\{c, d\}.$$

We distinguish between five cases (throughout the outer terms of the inequalities will pose a contradiction to one of the preceding three inequalities): (i) if $a = b$, $c = d$, then

$\min\{a,b\} = a < c = \min\{c,d\}$. Absurd. (ii) if $a \geq b$, $c \geq d$, then $a \geq b = \min\{a,b\} \geq min\{c,d\} = d$. Absurd. (iii) if $a \geq b$, $c \leq d$, then $a \geq b = \min\{a,b\} \geq \min\{c,d\} = c$. Absurd. (iv) if $a < b$, $c \geq d$, then $b > a = \min\{a,b\} \geq \min\{c,d\} = d$. Absurd. (v) if $a \leq b$, $c \leq d$, then $a = \min\{a,b\} \geq \min\{c,d\} \geq c$. Absurd. □

Proposition 8. *Suppose that upper contour sets $U_t(p)$ are a lattice for all p. Then the preference relation $\succsim_t^x$ satisfies reciprocity.*

Proof. There are two possible contrapositions to reciprocity: (i) there exist x_1, y_1 such that $y_1 < \underline{y}_t(x_1) \equiv y_2$, yet $x_1 < \underline{x}_t(y_1) \equiv x_2$. Or, this implies that $m_t(x_1, y_1) < m_t(x_1, y_2)$ and $m_t(x_1, y_1) < m_t(x_2, y_1)$. So $U_t(\min\{m_t(x_1, y_2); m_t(x_2, y_1)\})$ is not a lattice. (ii) there exist x_2, y_2 such that $y_2 > \overline{y}_t(x_2) \equiv y_1$, yet $x_2 > \overline{x}_t(y_2) \equiv x_1$. Or, this implies that, $m_t(x_2, y_2) < m_t(x_2, y_1)$ and $m_t(x_2, y_2) < m_t(x_1, y_2)$. So $U_t(\min\{m_t(x_1, y_2); m_t(x_2, y_1)\})$ is not a lattice. □

D.3 Characterization of reciprocity via $\mathcal{P}_t$

We now provide a proof of proposition 3. We prove each claim in turn.

Claim 1: $\mathcal{P}_t(y)$ is convex.

Proof. Fix $y_1 < y_2 < y_3$ such that $(x, y_1), (x, y_3) \in \Gamma_t$. We show that then also $(x, y_2) \in \mathcal{P}_t$. Due to convexity of $Y_t(x)$ as implied by weakly single-peaked preferences, this is readily the case when both $y_1, y_3 \in Y_t(x)$. Three cases remain.
(i) $y_1 \in Y_t(x)$ and $x \in X_t(y_3)$. If $y_2 \notin Y_t(x)$, then both $m_t(x, y_1) > m_t(x, y_2)$ and $m_t(x, y_3) > m_t(x, y_2)$. Suppose to the contrary that $(x, y_2) \notin \mathcal{P}_t$. Then both, $y_2 \notin Y_t(x)$ so that $m_t(x, y_1) > m_t(x, y_2)$ and $m_t(x, y_2) > m_t(x, y_2)$, and $x \notin X_t(y_2)$ so that either $x < X_t(y_2)$ or $x > X_t(y_2)$. In the former case $\underline{x}_t(y_2) > x$ and $m_t(x, y_2) < m_t(\underline{x}_t(y_2), y_2)$. Denoting $p \equiv \min\{m_t(x, y_3), m_t(\underline{x}_t(y_2), y_2)\}$, we have that $(x, y_3), (\underline{x}_t(y_2), y_2)$ belong to $U_t(p)$, but not (x, y_2). Then $U_t(p)$ is not a lattice. In the latter case $\overline{x}_t(y_2) < x$ and $m_t(x, y_2) < m_t(\overline{x}_t(y_2), y_2)$. Denoting $p \equiv \min\{m_t(x, y_1), m_t(\overline{x}_t(y_2), y_2)\}$, we have that $(x, y_1), (\overline{x}_t(y_2), y_2)$ belong to $U_t(p)$, but not (x, y_2). Once more, $U_t(p)$ is not a lattice. Having established, provided preferences are weakly single-peaked and satisfy weak single-crossing and reciprocity, that $U_t(p)$ is a lattice for all p, this poses the desired contradiction.
(ii) $y_3 \in Y_t(x)$ and $x \in X_t(y_1)$. Symmetric arguments as in (i) apply.
(iii) $x \in X_t(y_1)$ and $x \in X_t(y_3)$. If $y_2 \in Y_t(x)$, convexity is satisfied. If $Y_t(x) < y_2$, then for $y_1' = \overline{y}_t(x)$, $y_1' \in Y_t(x)$ and $x \in X_t(y_3)$, so that (i) establishes that $(x, y_2) \in \mathcal{P}_t$. If $Y_t(x) > y_2$, then for $y_3' = \underline{y}_t(x)$, $y_3' \in Y_t(x)$ and $x \in X_t(y_1)$, so that (ii) establishes that $(x, y_2) \in \mathcal{P}_t$. □

Claim 2: $\mathcal{P}_t$ is connected (this means that between any two pairs of types in $\mathcal{P}_t$, one can stay within $\mathcal{P}_t$ by moving along the horizontal, vertical axis or diagonal and connect the two pairs of types.

Proof. Consider two adjacent types $x_1 < x_2$. We show that $Y_t(x_1)$ and $Y_t(x_2)$ are connected. Since both $Y_t(x_1)$ and $Y_t(x_2)$ are convex due to weak single-crossing, it remains to consider two cases:
(i) Suppose that $\overline{y}_t(x_1) < \underline{y}_t(x_2)$. Then pick $y : \overline{y}_t(x_1) < y < \underline{y}_t(x_2)$. Then reciprocity

implies that $\underline{x}_t(y) \leq x_2$ and $\overline{x}_t(y) \geq x_1$, so that $\{x_1, x_2\} \in X_t(y)$ for all such y. In particular, it follows one can connect $(x_1, \overline{y}_t(x_1))$ and $(x_2, \underline{y}_t(x_2))$ via a path in $\mathcal{P}_t$.
(ii) Suppose that $\underline{y}_t(x_1) > \overline{y}_t(x_2)$. Then there exists $y : \overline{y}_t(x_2) < y < \underline{y}_t(x_1)$. But this is impossible according to single-crossing: if type x_1 strictly prefers $\underline{y}_t(x_1)$ over $\overline{y}_t(x_2)$, then type x_2 must at weakly share this and weakly prefer $\underline{y}_t(x_1)$ over $\overline{y}_t(x_2)$. But then $\underline{y}_t(x_1) \in Y_t(x_1)$ which poses the desired contradiction. □

Claim 3: $\mathcal{P}_t$ contains both $(x_{\max}, y_{\max})$ and $(x_{\min}, y_{\min})$

Proof. We show that $(x_{\max}, y_{\max})$ is in $\mathcal{P}_t$. That $(x_{\min}, y_{\min})$ is in $\mathcal{P}_t$ then follows along identical lines. Indeed, if $\overline{x}_t(y_{\max}) < x_{\max}$, reciprocity requires that $\overline{y}_t(x_{\max}) \geq y_{\max}$, so that $\overline{y}_t(x_{\max}) = y_{\max}$. □

D.4 Proof of theorem 4, non-stationary Shimer and Smith

We first prove lemma 4.

Proof. We proof the claim for $x_{\max}$, identical arguments apply for $x_{\min}$. Suppose to the contrary. Then there exists time t, state ω and type $y_j \in Y$ such that $f(x_{\max}, y_{\max}) - V_t^\omega(x_{\max}) - V_t^\omega(y_{\max}) < f(x_{\max}, y_j) - V_t^\omega(x_{\max}) - V_t^\omega(y_j)$. Or, applying the mimicking argument 3, this implies that there exists a measure $Q_t^\omega(x|y_{\max})$ whose sum is less than one such that

$$f(x_{\max}, y_{\max}) - f(x_{\max}, y_j) < V_t^\omega(y_{\max}) - V_t^\omega(y_j) \leq \sum_{x \in X} \big[f(x, y_{\max}) - f(x, y_j)\big] Q_t^\omega(x|y_{\max}).$$

Then supermodularity implies the first, the boundary condition the second inequality in what follows:

$$\begin{aligned}
\sum_{x \in X} \big[f(x, y_{\max}) - f(x, y_j)\big] Q_t^\omega(x|y_{\max}) &\leq \big[f(x_{\max}, y_{\max}) - f(x_{\max}, y_j)\big] \sum_{x \in X} Q_t^\omega(x|y_{\max}) \\
&\leq f(x_{\max}, y_{\max}) - f(x_{\max}, y_j).
\end{aligned}$$

This poses the desired contradiction. □

We next prove that $S_t(x, y)$ inherits Lipschitz-continuity from $f(x, y)$:

Lemma 8. *Suppose that output f is Lipschitz-continuous with constant L^f. Then, for any state and time, surplus S_t is Lipschitz-continuous with constant $3L^f$.*

Proof. Observe that

$$\begin{aligned}
&\big|S_t(x', y') - S_t(x'', y'')\big| \leq \big|f(x', y') - f(x'', y'')\big| + \big|V_t(x') - V_t(x'')\big| + \big|V_t(y') - V_t(y'')\big| \\
&\quad \leq \big|f(x', y') - f(x'', y'')\big| + \Big|\sum_y \big[f(x', y) - f(x'', y)\big] Q(y|x')\Big| + \Big|\sum_x \big[f(x, y') - f(x, y'')\big] Q(x|y')\Big|
\end{aligned}$$

due to the mimicking argument 3. □

We then provide a proof of the key lemma 5.

Proof. Pick arbitrary $(x_3, y_3) \in \overline{\mathcal{E}}_t$. Four cases are of interest. We show that in the first three cases $m_t(x_3, y_3) \geq \hat{p}_t$. First, if $(x_3, y_3) \in \mathcal{P}_t$, this is readily the case. Secondly, if $(x_3, y_3) \notin \mathcal{P}_t$, yet there exists y_0 such that $x_3 \in x_t(y_0)$, then monotonicity of the correspondence $y \mapsto x_t(y)$ due to single-crossing and $x_3 < \underline{x}_t(y_3)$ due to $(x_3, y_3) \in \overline{\mathcal{E}}_t$ imply that $y_0 < y_3$. Then weakly single-peaked preferences give that $m_t(x_3, \underline{y}_t(x_3)) \geq m_t(x_3, y_3) \geq m_t(x_3, y_0) \geq \hat{p}_t$ as desired. Thirdly, if $(x_3, y_3) \notin \mathcal{P}_t$, yet there exists x_0 such that $y_3 \in y_t(x_0)$, a symmetric reasoning applies.

Thus consider the fourth case: suppose that $(x_3, y_3) \notin \mathcal{P}_t$ and there do not exist x_0, y_0 such that $x_3 \in x_t(y_0)$ or $y_3 \in y_t(x_0)$. Consider, if well-defined, $y_1 \equiv \max\{y : \overline{x}_t(y_1) < x_3\}$ and $x_1 \equiv \max\{x : \overline{y}_t(x_1) < y_3\}$.

Since $(x_{\min}, y_{\min}) \in \mathcal{P}_t$, either $x_{\min} = \underline{x}_t(y_{\min})$ or $y_{\min} = \underline{y}_t(x_{\min})$. Since there do not exist x_0, y_0 with the desired properties, neither $y_3 \in y_t(x_{\min})$, nor $x_3 \in x_t(y_{\min})$. It follows that either $\{y : \overline{x}_t(y_1) < x_3\}$ or $\{x : \overline{y}_t(x_1) < y_3\}$ is non-empty (for $y_{\min}$ or $x_{\min}$ respectively belong to it). Thus x_1 or y_1 (but not necessarily both) is well-defined. Thus, without loss of generality, take x_1 to be well-defined

Then define $x_2 \equiv \min\{x : \underline{y}_t(x_2) > y_3\}$. Since $(x_3, y_3) \in \overline{\mathcal{E}}_t \setminus \mathcal{P}_t$, $y_3 < \underline{y}_t(x_3)$. It follows that $\{x : \underline{y}_t(x_2) > y_3\}$ is non-empty (for x_3 belongs to it), so that x_2 is well-defined. By construction, $x_1 < x_2 \leq x_3$ and x_1, x_2 are adjacent.

Next, notice that $x_3 < \underline{x}_t(y_3)$ in conjunction with weak single-peaked preferences implies that $m_t(x_3, y_3) \geq m_t(x_2, y_3) \geq m_t(x_2, y_1)$. Notice further that $m_t(x_1, y_1) \geq \hat{p}_t$. It thus suffices to prove that $m_t(x_2, y_1) \geq m_t(x_1, y_1) - \epsilon^N$.

This follows from the fact that $f(x, y)$ is Lipschitz-continuous. For then, according to the preceding lemma $S_t(x, y)$, is Lipschitz-continuous with constant $\frac{q}{r}\frac{3}{\underline{N}}$. And therefore, since x_1 and x_2 are adjacent, $|S_t(x_2, y_1) - S_t(x_1, y_1)| < 3\frac{q}{\underline{N}}$. Therefore, $m_t(x_2, y_1) = 1 - \Xi_t(S_t(x_2, y_1)) \geq 1 - \Xi_t(S_t(x_1, y_1) + 3\frac{q}{\underline{N}}) = m_t(x_1, y_1) + \Xi_t(S_t(x_1, y_1)) - \Xi_t(S_t(x_1, y_1) + 3\frac{q}{\underline{N}})$. Since Ξ_t is Lipschitz-continuous with constant $\frac{L^\xi}{3q}$, it follows that $\Xi_t(S + 3\frac{q}{\underline{N}}) - \Xi_t(S) \leq \frac{L^\xi}{\underline{N}}$ for all S. Then the preceding implies that $m_t(x_3, y_3) \geq m_t(x_2, y_1) \geq \hat{p}_t - \frac{L^\xi}{\underline{N}}$, as desired. □

D.5 Self-acceptance at the boundary: steady state

We here provide a result which is interesting in its own right. Unlike the main text, we consider here an identical environment as in Shimer and Smith (2000). The economy is stationary and there are no pair-specific production shocks. We show that Shimer and Smith's boundary condition can be relaxed in the steady state: rather than imposing that output for lowest types is non-increasing, we add the complementarity condition that output is log supermodular.

Proposition 9. *Suppose that the economy is at the steady state, and populations are symmetric. If f is supermodular and log supermodular, then the lowest type self-accepts.*

To achieve this we construct an upper bound on the stationary value of search which does not depend on other agent types' value of search. We then prove that under given conditions this upper bound is smaller than the payoff the highest and lowest types can guarantee themselves by matching with an agent of equal type. In this proof, we denote $M(k|x)$ the expected discounted probability that a type x with a type k. For ease of exposition, we denote $x_{min} = 1$ and $x_{max} = N$. In the steady state, the value of search of agent type x can be expressed as such:

$$V(x) = \sum_{k=1}^{N} \frac{1}{2}\big(f(x,k) + V(x) - V(k)\big)M(k|x).$$

Isolating $V(x)$ this gives

$$V(x) = \frac{1}{2-\bar{M}_x} \sum_{k=1}^{N} \big(f(x,k) - V(k)\big)M(k|x),$$

where $\bar{M}_x = \sum_{k=1}^{N} M(k|x)$. Then by applying the mimicking argument we obtain the following upper bound on $V(x)$:

$$V(x) \leq \frac{1}{2-\bar{M}_x}\big[\sum_{k=1}^{N} f(x,k)M(k|x) \quad - \quad \sum_{k=1}^{N}\sum_{h=1}^{N} \pi(h,k)M(h|x)M(k|x)\big].$$

Then set

$$V(x) \leq \bar{V}(x) \equiv \max_{\{M(k|x)\}_{k=1}^{N}} \frac{1}{2-\bar{M}_x}\big[\sum_{k=1}^{N} f(x,k)M(k|x) \quad - \quad \sum_{k=1}^{N}\sum_{h=1}^{N} \frac{f(h,k)}{2}M(h|x)M(k|x)\big],$$

where the max is taken over discounted probabilities $\{M(k|x)\}_{k=1}^{N}$ which must be non-negative and sum to less or equal to one, i.e. $\sum_{k=1}^{N} M(k|x) \leq 1$. Given optimal discounted probabilities $\{M(k|x)\}_{k=1}^{N}$, shifting mass between two types m and n to which one has assigned positive discounted probability can not raise $\bar{V}$. Or, let $\hat{M}^{\alpha}(n|x) = \alpha M(m|x) + M(n|x)$, $\hat{M}^{\alpha}(m|x) = (1-\alpha)M(m|x)$ and $\hat{M}^{\alpha}(k|x) = M(k|x)$ otherwise. And let

$$v(\alpha) = \sum_{k=1}^{N} f(x,k)\hat{M}^{\alpha}(k|x) \quad - \quad \sum_{k=1}^{N}\sum_{h=1}^{N} \frac{f(h,k)}{2}\hat{M}^{\alpha}(h|x)\hat{M}^{\alpha}(k|x).$$

Then due to the optimality of $\{M(k|x)\}_{k=1}^{N}$, it must be that $v(0) \geq v(\alpha)$ for all $\alpha \in [0,1]$, and therefore $v'(0) \leq 0$. Since the roles of m and n are interchangeable, this implies that $v'(0) = 0$. Differentiating gives

$$0 = v'(0) = f(x,n) - f(x,m) - \sum_{k=1}^{N}[f(k,n) - f(k,m)]M(k|x).$$

Or, when $x = 1$ and $n < m$ this gives

$$f(1,n) - f(1,m) = \sum_{k=1}^{N}[f(k,n) - f(k,m)]M(k|x) < [f(1,n) - f(1,m)]\sum_{k=1}^{N} M(k|x)$$

where the latter inequality follows from supermodularity. This is impossible since $\sum\limits_{k=1}^{N} M(k|x) \leq 1$. This establishes that for $x = 1$ the maximizing weights are zero for all but a single type k. Likewise, when $x = N$ and $n > m$ this gives

$$f(N,n) - f(N,m) = \sum_{k=1}^{N}[f(k,n) - f(k,m)]M(k|x) < [f(N,n) - f(N,m)]\sum_{k=1}^{N} M(k|x),$$

where the latter inequality follows once more from supermodularity. This poses the desired contradiction. In conclusion we have shown that

$$\bar{V}(1) = \frac{\bar{M}_1}{2-\bar{M}_1}\left[f(1,y) - \bar{M}_1\frac{f(y,y)}{2}\right] \quad \text{and} \quad \bar{V}(N) = \frac{\bar{M}_N}{2-\bar{M}_N}\left[f(1,y) - \bar{M}_N\frac{f(y,y)}{2}\right]$$

for some agent type y (not the same for $x = 1$ and $x = N$). In what remains we derive necessary and sufficient conditions that ensure both $\bar{V}(1) \le \frac{f(1,1)}{2}$ and $\bar{V}(N) \le \frac{f(N,N)}{2}$.

Notice that for both $x \in \{1, N\}$ the bound $\overline{M}_x \mapsto \overline{V}(x)$ is weakly convex if $f(x,y) \geq f(y,y)$ and strictly concave otherwise. If it is convex, the maximizing $\overline{M}_x$ must lie on the boundary $\{0,1\}$. If it is concave, the maximizing $\overline{M}_x$ is characterized by, if interior, the first-order condition. Or,

$$\overline{M}_x = \begin{cases} 1 & \text{if } \frac{f(y,y)}{f(x,y)} \le \frac{4}{3} \\ 2 - \frac{2\sqrt{f(y,y)(f(y,y)-f(x,y))}}{f(y,y)} & \text{if } \frac{f(y,y)}{f(x,y)} > \frac{4}{3}. \end{cases}$$

If $\overline{M}_x = 1$, then supermodularity implies that the $\overline{V}(x) \le \frac{f(x,x)}{2}$, all but ensuring self-acceptance. (The same upper bound holds trivially for $\overline{M}_x = 0$.)

Thus focus on the case where $\frac{f(y,y)}{f(x,y)} > \frac{4}{3}$. Then

$$\overline{V}(x) = 2f(y,y) - f(x,y) - 2\sqrt{f(y,y)(f(y,y) - f(x,y))}$$

We then seek to contradict the claim that x's option value of search may exceed $\frac{f(x,x)}{2}$, thereby ensuring that she will match with her own type.

To simplify the algebra, let $\gamma = f(y,y)/f(x,y)$, and $\theta = f(x,y)/f(x,x)$. Then $\overline{V}(x) = h(\gamma)y$ where h is defined as $h(\gamma) = 2\gamma - 1 - 2\sqrt{\gamma(\gamma-1)}$ and $\gamma \geq \frac{4}{3}$. Then $\overline{V}(x) \le \frac{f(x,x)}{2}$ if and only if $h(\gamma) \le \frac{1}{2\theta}$, or equivalently

$$\theta \le \frac{1}{2h(\gamma)} \quad \Leftrightarrow \quad \frac{f(x,y)}{f(x,x)} \le \frac{1}{2h(\frac{f(y,y)}{f(x,y)})}.$$

Now observe that

$$\frac{f(y,y)}{f(x,y)} < \frac{1}{2h(\frac{f(y,y)}{f(x,y)})},$$

which means that log supermodularity is sufficient, but not necessary to achieve this.

D.6 Counter-example

Example (Horizontal differentiation $\not\Rightarrow$ reciprocity)**.** *Consider an economy with two symmetric populations, in the steady state, with three types $x_3 > x_2 > x_1$, and negligible pair-specific shocks. The supermodular match output, $f(x,y)$, exhibits horizontal differen-*

tiation, and is given by the following matrix:

f	x_1	x_2	x_3
x_3	0	9	10
x_2	9	10	9
x_1	10	9	0

Suppose there are extremely few agents of type x_1 and x_2 in the search pool, so the probability of meeting them is close to zero. Furthermore, suppose that the discounted expected probability of meeting x_3 is equal to $1/2$. In such case, the medium type x_2 prefers to meet x_1 over himself, upsetting reciprocity.

Proof. Denote q the discounted expected probability of meeting x_3. We guess (then we will verify) that, upon meeting, x_3 matches with everyone but not x_1; and that x_2 prefers to match with x_1 than x_2:

$$S(x_3, x_1) < 0 \text{ and } S(x_3, x_2) > 0 \text{ and } S(x_1, x_2) > S(x_2, x_1)$$

Accordingly, the value of search of each type is given by:

$$\begin{aligned}
V(x_1) &= 0 \\
V(x_3) &= q\frac{f(x_3, x_3)}{2} \\
V(x_2) &= \frac{q}{2-q}(f(x_3, x_2) - \alpha\frac{f(x_3, x_3)}{2})
\end{aligned}$$

First, observe that $S(x_2, x_3) > 0$ is equivalent to $V(x_2) > 0$. So it's trivially satisfied when q and $f(x_3, x_3) - f(x_3, x_2)$ are not too high (it is satisfied in our case). Then, using the expression of the value of search, $S(x_3, x_1) < 0$ and $S(x_1, x_2) > S(x_2, x_2)$, can be expressed as such:

$$\begin{aligned}
\frac{q}{2-q}(f(x_3, x_2) - q\frac{f(x_3, x_3)}{2}) &> f(x_2, x_2) - f(x_2, x_1) \\
f(x_3, x_1) &< q\frac{f(x_3, x_3)}{2}
\end{aligned}$$

We can now plug the value of q and f, and check that these two inequalities are satisfied, and so our conjectured equilibrium is indeed an equilibrium. □

E Related Literature

E.1 Theoretical results on sorting

Table as in our first paper.

Interest in assortative matching extends beyond matching with random search. We mention here two prominent papers in related frameworks that derive conditions for PAM. Legros and Newman (2007) show that a generalized increasing differences condition extends Becker (1973) to not fully transferable payoffs. Studying directed search, Eeckhout and Kircher (2010) show that positive assortative matching obtains if and only if the joint match surplus is root supermodular.

1. Connection with NTU

TU Conditions on f = NTU conditions on $\triangle f$.

2. Shimer and Smith (2000) and theorem 3.

If search frictions upset PAM, why does a set based notion of said concept obtain in Shimer and Smith (2000)? The short answer is that match outcomes, i.e., meeting-contingent matching patterns within a deterministic framework do not fully reflect preferences over such. As a consequence, a more rudimentary notion of reciprocity (self-preferences of the lowest and highest types[25]) will be sufficient to satisfy Shimer and Smith (2000)'s set-based notion of PAM over outcomes: matching sets which are increasing in types. This is not to say that preferences satisfy PAM. Our characterization of the defining properties of preferences—single-crossing, single-peakedness, yet not reciprocity —remains true in a stationary model without pair-specific production shocks.

Nevertheless, we recover a version of Shimer and Smith (2000)'s original sorting result and extend it to non-stationary environments with probabilistic matching (theorem 4). Under identical complementarity and boundary conditions as theirs, we prove that there is PAM for the subset of pairs that match with probability less than the min max meeting-contingent match probability $\min_x \max_y m_t(x, y)$, i.e., the outer set. This encompasses Shimer and Smith's original result: owing to both the degeneracy of match probabilities and the stationarity assumption, the min max-probability in their model is one—if it were zero for some type, then such type would never match, yielding a zero option value of search. Thus PAM obtains for all types. More generally, our result establishes that with negligible pair-specific production shocks matching patterns exhibit PAM even in non-stationary environments (provided that search frictions are sufficiently large, so that no agent type can afford to reject all other agent types). As before, this is possible because $m_t(x, y)$ does not represent preferences over meetings.

[25]As a by-product of our analysis, we show that the sufficient conditions of Shimer and Smith can be weakened. To obtain PAM, they rule out strict vertical differentiation by assuming that the lowest type is more productive when matched with another low type, that is $y \mapsto f(x_{min}, y)$ is decreasing in y. We show that log supermodularity of f, as implied by their conditions under vertically differentiated types, renders their boundary condition obsolete in the steady state.

E.2 Theoretical results on existence and uniqueness

Search and matching

Existence proofs feature prominently in the literature on matching with random search, with the exclusive focus being on the existence of steady state equilibria. Shimer and Smith (2000) prove existence of a steady state equilibrium under Nash bargaining and quadratic search. Smith (2006) derives an identical result when payoffs are not transferable. Lauermann and Nöldeke (2015) and Lauermann et al. (2020) generalize this literature to a broader class of meeting rates. While in Bonneton and Sandmann (2019) we had provided the first existence proof in the non-stationary NTU framework, we here present the first non-stationary *uniqueness* result in the theory of random search matching. Unlike in our earlier work we consider a stochastic environment.

Our uniqueness result casts doubt on the robustness of multiple self-fulfilling equilibrium paths frequently reported in the literature (as for instance in Diamond (1982), Boldrin et al. (1993), Burdett and Coles (1997), Eeckhout and Lindenlaub (2019)). As a common theme, multiple equilibrium paths arise because agents can perfectly foresee and coordinate their actions.[26] In our paper, random entry fosters aggregate uncertainty and thereby obstructs self-fulfilling equilibrium paths. Intuitively, this arises because starting from a given size and composition of the search pool, its evolution is not foreseeable. It follows (unlike in a deterministic framework) that the value of search is continuous in the size and composition of the search pool. This implies that if acceptance thresholds were unduly low along one particular path, they must be low across all neighboring paths, too. Further, if acceptance rules are dominated along some paths, this narrows the number of rationalizable acceptance threshold rules along neighboring paths. Proceeding iteratively, we can eliminate dominated acceptance threshold rules across neighboring paths, ultimately converging to a unique acceptance threshold rule (and thereby implied value of search).

The idea that noise, however small, can break the perfect foresight property and thus restore equilibrium uniqueness was first presented in Frankel and Pauzner (2000). [...]

2. Delarue (2002)

Delarue (2002)'s theorem is remarkable. It synthesises various techniques known to establish the unique existence of a solution to systems of FBSDEs. First, drawing on a fixed-point argument, he establishes the unique existence of a solution in short intervals of time. The PDE representation of (smooth) FBSDEs affords bounds on this solution, ruling out explosive behavior. He thus employs an induction argument and extends the initial solution to time intervals of arbitrary length.

Open questions: infinite horizon, infinite dimensional system corresponding to a continuum of types, stability.

[26] For instance, consider the quadratic search technology where the expected number of meetings rises in the mass of agents searching. Here, increased exit of types x decreases the expected number of meetings and thus the value of search for types y. This, in turn, fosters less selective acceptance rules (and greater exit) for agents types y, too. Such feedback loop may give rise to multiple equilibria, whereby agents coordinate on more or less selective match acceptance strategies.

E.3 Pair-specific production shocks in the literature

The inclusion of pair-specific production shocks is commonplace in empirical work on matching. It sometimes enters as measurement error.

1. Cite Choo and Siow, Chiappori et al.

In Choo and Siow, pair-specific production shocks and measurement error the same. With search frictions not true. Suppose one were to observe wage data generated by a model including said shocks. And suppose one had correctly identified the ranking of individual types (following identical arguments as in Hagedorn et al. (2017)). Then if one were to attribute dispersion in wages to measurement error rather than pair-specific production shocks which vary across pairs, estimates of the production function would be biased.

2. Chade

The case for pair-specific production shocks

In our view, pair-specific production shocks are essential to our understanding of matching data.

First, it is known that the unemployment rate at the beginning of an unemployment spell correlates negatively with the contemporary wage (refer to Bils (1985) and Beaudry and DiNardo (1991)). Hagedorn and Manovskii (2013) show empirically (alas without controlling for unobserved heterogeneity) that various variables summarizing past aggregate labor market conditions have explanatory power for current wages only because they are correlated with the sampling rate of production shocks. They lose any predictive power once pair-specific production shocks are accounted for. Quite intuitively, less search in times of high unemployment rates leads to less sampling of production shocks and thus less surplus creation and lower wages.

Secondly, there is evidence from a calibrated version of the stationary random search model due to Shimer and Smith (2000) with on-the-job search. Lopes de Melo (2016) shows that said calibration provides a good fit of key moments of matching data, but falls short on capturing the dispersion of firm fixed-effects in the standard Abowd et al. regression on log wages.[27] This dispersion is low in the calibrated model, because wages are not increasing in firm productivity; compared to firm y_1, firm y_2 may pay a greater wage to x_2, but a lower wage to x_1. Reminiscent of here-considered pair-specific production shocks, De Melo (2015) suggests that stochastic worker types increase the

[27] Using longitudinal matched employer–employee data, Abowd et al. estimate firm and worker fixed effects of an otherwise standard wage regression (refer to de Melo (2015), table 1 for a snythesis of results across various studies). Initially, they argued that the estimate of fixed effects of the regression corresponds to workers' and firms' productivity, accounting for unobserved heterogeneity. According to this argument, the covariance between worker and firm fixed effects would thus identify sorting. This approach has since been discredited. The reason is that standard search and matching models predict that a given worker's expected wage is non-increasing in firm productivity (more specifically single-peaked under known complementarity conditions as we show in this paper). This is at odds with a key identifying assumption in Abowd et al. which requires that wages be monotone increasing in firm productivity (refer to Gautier Teuling (2006) and Eeckhout and Kircher (2011) for the initial finding, footnote 2 in Hagedorn, Law, Manovskii (2016) for further empirical corrobation, and de Melo (2015) for an in-depth discussion of implications for key moments).

dispersion in firm fixed effects in a calibrated model.

Thirdly, the random search matching model without pair-specific production shocks predicts that the time t flow of match creation between two types x, y, denoted $\zeta_t(x, y)$ satisfy a constant ratio property: whenever $\zeta_t(x, y)$ is non-zero for combination of pairs x_1, x_2 and y_1, y_2, it must satisfy $\frac{\zeta_t(x_2,y_2)}{\zeta_t(x_2,y_1)} = \frac{\zeta_t(x_1,y_2)}{\zeta_t(x_1,y_1)}$.[28] This prediction is difficult to assess empirically, for characteristics observable to the econometrician very incompletely describe workers' heterogeneity. Individual types are for the most part unobservable. Hagedorn, Law, Manovskii (2016) show however that unobserved heterogeneity and matching decisions in Shimer and Smith's standard random search and matching model can be identified. Figure 6 of their work plots their estimates of $\zeta_t(x, y)$ for German data. Their empirical strategy identifies each individual with a unique type, workers' types, so that $\mu(x)$ is uniformly distributed and their figure 6 is proportional to both the flow rate of matching and an empirical estimate of $\mu(y)m(x, y)$. Visually, the ratio property does not obtain, rendering their estimates inconsistent with a stationary model without pair-specific production shocks. Reported estimates do satisfy single-peakedness of $x \mapsto \mu(y)m(x, y)$ however, as is implied by our theorem 2 and consistent with a model in which there are pair-specific production shocks.

[28]The prediction that the flow rate of match creation neither depends on the functional form of the meeting rate other than that it satisfy anonymity, nor on stationarity of the economy. To see this, consider the flow rate of match creation of two types x, y in detail: $\zeta_t(x, y) \equiv \beta_t \mu_t(y)\mu_t(x)m_t(x, y)$, where β_t is a time-varying constant governing the speed of meetings, $\mu_t(y)$ is the mass of agent types searching, and $m_t(x, y)$ is the match indicator function, which is zero or one absent pair-specific production shocks.

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