Repeated Games in Self-Sustained Communities

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Abstract

In the current mobile world, repeated relationships ("communities") must be self-sustained. We formulate a framework in which some or all players strategically choose whether to terminate or repeat an N-person game. A dynamic game ends when a certain number of players choose termination. To sustain the maximal set of outcomes while maintaining the community, the players may need to end the interaction as an equilibrium punishment. However, since termination is an absorbing state, players cannot "reward" one another for the appropriate use of termination. We show that voluntary termination is always incentive-compatible when players vote on game continuation/termination simultaneously under non-unanimous ending rules. When voting is sequential, we construct a new mechanism to make termination incentive-compatible. The frontier of equilibrium outcomes can be larger than that of ordinary repeated games without the termination option, but, in that case, the set of equilibrium outcomes expands in the direction of unequal payoffs. (150 words)

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1 Introduction

1.1 Motivation

In traditional communities (villages, churches, etc.), the members do not change for a long time. Such long-run relationships of fixed members can be viewed as (infinitely) repeated games (e.g., Friedman, 1971, and Aumann and Shapley, 1976).[1] The other traditional interactions of long-lived agents are spot transaction meetings in marketplaces. These one-shot relationships fit well into the model of random-matching games (e.g., Rosenthal, 1979, Kandori, 1992a, and Ellison, 1994). As the world had become increasingly volatile so that economic agents can change locations and even names to strategically change partners, models are emerging to formulate interactions whose durations are determined endogenously. An important class is such that randomly matched partners can choose to continue the partnership by mutual agreement and, if they choose to separate, they can find a new partner in the matching pool, as transactions in the Internet (e.g., Ghosh and Ray, 1996, and Fujiwara-Greve and Okuno-Fujiwara, 2009). However, all of the above models assume that the horizon of the community is exogenous.

In this paper we consider communities whose durations are endogenous. For example, the duration of organizations, firm cartels, and trade agreements are strategically determined by the members, and termination destroys the group entirely. There is a variety of possibilities how the duration of a community is endogenously chosen (e.g., voting, bankruptcy) and how the players can influence the continuation payoffs after the community termination. To focus, we consider the framework with the following features: Termination is determined by some or all players' votes, and it is an absorbing state in which each player receives an exogenous termination payoff that is independent from the history within the game.[2] This framework thus requires that incentives must be devised within the self-sustained community. We also allow two kinds of players, those who can vote on the game duration ("mobile players") and those who cannot ("immobile players"), to investigate the effect of the voting right structure on the equilibrium outcomes.

An important motivation to consider long-run players is to enforce non-myopic actions. In

[1] If the horizon is known to end at some point, such as interactions in a school, finitely repeated games (e.g., Benoit and Krishna, 1985) are the appropriate model.

[2] We can allow that some of the players may interact in the future, provided that they cannot change the payoffs in future games based on the history of the terminated community. For example, future games may involve new players who cannot know the history of the terminated community.

models with an ever-lasting community, it was shown that non-myopic actions are sustainable by constructing punishments, thanks to the models' feature that there is always a future interaction. For example, the key contribution of Fudenberg and Maskin (1986) was to include a reward phase for punishers when they successfully play a min-maxing action profile against a deviator for sufficiently many periods, even though min-maxing is not a one-shot Nash equilibrium. In our model, there are only two strategy combinations that can become the severest punishment[3]: terminating the game to force the punished player to receive an exogenous termination payoff or min-maxing the deviator while continuing the game.[4] However, players cannot reward one another after "cooperating" to terminate the game as a punishment. Making game-termination credible is a new theoretical challenge. Moreover, the appropriate reservation level is dependent on how a player can influence community termination. Hence we need an extended notion of the minmax point to establish the scope of sustainable payoff combinations.

When mobile players vote simultaneously, in most cases they can enforce termination without reward. Therefore, we can prove an extended folk theorem by an analogy of Fudenberg and Maskin (1986). The logic is that, if sufficiently many other mobile players choose to terminate the game, no single player can change the outcome, and thus it is a weak best response to choose termination. We cannot use this logic when unanimity among mobile players is required to terminate the community. In that case, each mobile player possesses veto power not to end the game and may use the power if she/he prefers continuing the game to terminating it. We identified environments in which termination can be enforced under the unanimity requirement, except when there is only one mobile player.

The most difficult case is when (i) there is a unique mobile player, (ii) termination is needed to punish someone, but (iii) the mobile player's termination payoff is less than her/his stage-game minmax value. For this case, we extend the notion of the effective min-max payoff proposed by Wen (1994) to give a modified folk theorem.

Even when termination is possible under simultaneous-voting, sequential voting can unravel, when some mobile player is pivotal and does not want a termination. We construct a new incentive mechanism to make players agree to terminate the game under sequential voting, which is similar to the one employed at some Japanese companies: if a mobile player who is

[3]This is the key to determining whether an outcome is sustainable, see Abreu (1988).

[4]Hence we provide an equilibrium foundation for the use of exit and voice in Hirschman (1972) as substitute disciplining devices.

supposed to vote for termination voted for continuation, then she would not be allowed to escape from the game anymore and receives the stage-game minmax value.[5] Thus, even though the reward is not possible after community termination, punishment for some mobile players may be possible to motivate them to choose termination.

In sum, we provide a foundation to the wisdom of repeated games that feasible and (appropriately-extended) individually rational payoff combinations are sustainable, even if the community can be terminated at will. At the same time, we show what kind of voting rules can make termination credible without future interactions. From the perspectives of mechanism design, if a community can choose among majority-based rules, the rules which are likely to expand the set of sustainable outcomes are non-unanimous and non-unilateral ones. Under such termination rules, mobile players can jointly enforce termination and min-maxing whichever is the severest punishment, for each deviator.

1.2 Literature

To place our paper, the dynamic models in which the length of interaction is endogenous can be divided into the "termination" models like this paper and the "recurrent" models mentioned in Section 1.1, where players stay in an ever-lasting community but form endogenous-length partnerships. There are not so many termination models in which the incentive must be created within the voluntary interaction (this paper and Casas-Arce, 2010).[6] Casas-Arce (2010) analyzes a principal-agent type repeated game, where the principal, who stays in the game until the end, can replace an agent (from a homogenous pool of agents), while an agent can quit as well. His folk theorem (Proposition 1) is due to the fact that all agents are "mobile".

In the recurrent models[7], once an interaction is terminated, players are randomly assigned a new partner and restart another voluntary interaction without information flow. Hence such models endogenize the termination payoffs, while termination payoffs are parameters of our

[5]In Japan, it is not easy to fire a permanent employee. If a firm needs to induce an employee to quit but she/he does not agree, a firm must allow her/him to stay but can assign her/him a meaningless task for a long time, which is called to put in "an expulsion room" or *oidashi-beya* (this term is coined by Asahi Shimbun, according to the article "No room for subtleties when laying off workers" of Japan Times, Jan. 26, 2013).

[6]Wilson and Wu (2017) considers a repeated Prisoner's Dilemma with termination payoffs and has a prediction of the set of payoff combinations (their Figure 1) with undominated strategies. But they do not prove that the set is sustainable by some equilibrium.

[7]The literature of recurrent models without incomplete information includes Datta (1996), Kranton (1996), Fujiwara-Greve and Okuno-Fujiwara (2009), and Immorlica, Lucier, and Rogers (2014).

model. In the recurrent models, the innovation is how to construct community punishments embedded in the strategy combination. Our contribution is to find out when a self-sustained community can use termination credibly.

Our termination payoffs can be called outside options, but our model is different from those in which players take an outside option to be inactive during a dynamic game (e.g., Hörner, 2002, and Halac, 2012). Such models do not have an absorbing state like ours, and thus one can add a reward phase to motivate players to be inactive if that is needed.

There are also related models in the industrial organization literature that have endogenous game termination. Rosenthal and Rubinstein (1984) do not use subgame-perfect equilibrium in their ruin game, and thus no "on-path" and "punishment" structure is considered. In the R&D model by Furusawa and Kawakami (2008) and in oligopoly models by Wiseman (2017) and Beviá et al. (2020), firms may go out of business endogenously. However, exit itself is not a strategic choice, and players indirectly control their survival and rival's exit through R&D activities, production actions, and so on. Therefore, making the direct choice of game-termination incentive-compatible is not an issue in these oligopolistic models.

A closer applied paper to ours is by Fuchs and Lippi (2006), who consider a two-country model where each country can strategically choose independent policy-making or a monetary union, not assuming that independence is always bad. They show that, in some cases, termination of the monetary union is optimal. In their model, after the monetary union is strategically terminated, the two countries enter a repeated game as independent policy-makers, and it is an important part of the design of an optimal equilibrium how to choose the punishment equilibrium in the game of independent policy-makers. Hence, they endogenize the "termination payoffs after exiting from the monetary union", but these are not the same as our termination payoffs because the dynamic game of the two countries itself never ends. Also, since the Folk Theorem holds for the repeated game of independent policy-makers, it is possible to reward players after optimal termination of the monetary union. By contrast, our point is that it may not be possible to reward players to terminate the game itself when that is needed.

Experimental research is also emerging for repeated games with termination options (Wilson and Wu, 2017, and Gaudeul et al., 2017). Wilson and Wu (2017) is closest to ours (their setup is that all players are mobile, and the stage game is Prisoner's Dilemma with imperfect public monitoring) and found that termination and min-maxing are appropriately used, indicating that real subjects take into account the extended notion of the minmax values.

This paper is organized as follows. Section 2 introduces the model and the key concepts as well as characterizes the extended minmax value for all majority-based termination rules. In Section 3, we show sufficient conditions for termination and min-maxing while continuing the game to be equilibrium punishments. Section 4 gives the main results of folk and anti-folk theorems for simultaneous-voting. Section 5 analyzes the most difficult case of the unanimous-ending rule to incentivize punishments. In Section 6, we consider sequential voting and introduce the "expulsion-room" mechanism. Section 7 concludes.

2 Model and Key Concepts

2.1 Repeated games with endogenous termination

Let $\{1,\ldots,N\} = \mathcal{N}$ be the set of players with a generic element n (sometimes j is also used). Throughout the paper, we assume that $N \geqq 2$. The time horizon is discrete and written as period $t = 0, 1, \ldots$. As long as the game continues, in each period, the same simultaneous-move game (the "stage game")

$$g^{st} : A_1 \times \cdots \times A_N \to \mathbb{R}^N$$

is played, where A_n is the set of actions available to player $n \in \mathcal{N}$ with a generic element a_n. We assume that (i) for each $n \in \mathcal{N}$, A_n is a compact and convex subset of $\mathbb{R}^{\ell_n}$ for some $\ell_n < \infty$, and (ii) g^{st} is a continuous function.[8]

In each period, before the stage game[9], a subset $\{1, 2, \ldots, M\} =: \mathcal{M} \subset \mathcal{N}$ of players (where $1 \leqq M \leqq N$)[10] vote either "continue" (action 0) or "end" (action 1). This stage can be interpreted as a communication stage among the players in $\mathcal{M}$. In the base model, we assume simultaneous voting to treat all players in $\mathcal{M}$ symmetrically except their payoff functions. (In

[8]We can alternatively start with a finite action game $h : S_1 \times \cdots \times S_N \to \mathbb{R}^N$ (where $|S_n| < \infty$ for each $n \in \mathcal{N}$) and let $A_n = \Delta S_n$ and $g^{st} = E[h]$. Then assumptions (i) and (ii) are satisfied. However, correlations over $\times_{n=1}^N \Delta S_n$ involves a subtle issue of measurability. Hence with this interpretation we only deal with $\Delta(\times_{n=1}^N S_n)$ as the correlated action profiles. There is also an issue of observability of mixed actions, see footnote 13.

[9]This order of decision making is to simplify the payoff sequence computation. For example, the immediate-ending equilibrium (see Lemma 3) gives each player on average the termination payoff. If the stage game is played first and then continuation decision is made in each period, then the average payoff of the same equilibrium is the average of one period of some Nash equilibrium payoff and the termination payoff, which only complicates the payoff sequence structure and does not give a new economic insight.

[10]The ordinary repeated game is the case when $M = 0$.

Section 6, we consider a sequential voting model.) We refer to players in $\mathcal{M}$ as **mobile players** and the rest as **immobile players**. The size M of mobile players is a parameter of the dynamic game.

Whether the game continues or ends is determined by the **θ-majority rule** such that the game ends if and only if at least $\theta \in \{1, \dots, M\}$ of mobile players choose "end", or

$$\sum_{m=1}^{M} z_{m,t} \geqq \theta,$$

where $z_{m,t} \in \{0, 1\}$ is the continue/end choice by player m at the beginning of period t. In particular, when $\theta = M$, we call it the **unanimous ending rule**, i.e., all mobile players (not necessarily all players) need to choose "end" in order to terminate the game.

The name "mobile players" is easily understood when there are only two players and one of them can unilaterally walk away from the repeated game to take an outside option (i.e., $\theta = 1 = M$). For general N-player games with immobile players, the mobile ones can be interpreted as senior members of an organization/team who can jointly (but non-cooperatively) choose whether to terminate the organization/team or not. Many partnership-type firms (law firms, accounting firms) can be endogenously terminated this way. There are also many prominent examples where all players are mobile, such as colluding firms, co-authors, and married couples.

We assume that, once the game ends, each player $n \in \mathcal{N}$ receives x_n *on average*, which is exogenously given and does not depend on any history in the game.[11] In other words, termination is the absorbing state that comes with a fixed outside option for each player. We call x_n the **termination payoff** for player n. It can be the average payoff of some equilibrium in a game after the current game ends. Figure 1 illustrates the timeline of the dynamic game.

Players use a common discount factor $\delta \in (0, 1)$ to evaluate an infinite sequence of one-shot payoffs. That is, if $(a_{1,t}, \dots, a_{N,t}) \in A_1 \times \cdots \times A_N$ is the vector of stage game actions played in period t and the game is terminated in period τ, then player n's total payoff is

$$\sum_{t=0}^{\tau-1} \delta^t g_n^{st}(a_{1,t}, \dots, a_{N,t}) + \frac{\delta^\tau}{1-\delta} x_n,$$

[11]The assumption of history-independent payoff after termination is also standard in the relational contract models. See Levin (2003) for example. Fujiwara-Greve and Yasuda (2011) analyze the repeated Prisoner's Dilemma with endogenous termination in which termination payoffs are history-independent but stochastic.

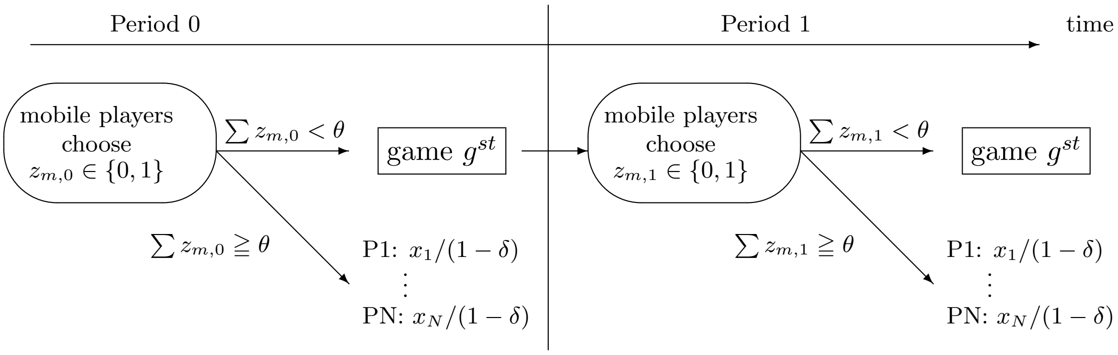

Figure 1: The repeated game with endogenous termination

and her average payoff is

$$(1-\delta)\sum_{t=0}^{\tau-1}\delta^t g_n^{st}(a_{1,t},\ldots,a_{N,t}) + \delta^\tau x_n.$$

We assume the standard perfect monitoring; all players observe realized (pure) actions in g^{st} taken by all players at the end of each period. Without loss of generality, we assume the existence of a public randomization device to correlate the stage-game action profiles, since we can approximate the average payoff vector of any correlated action profile by an infinite sequence of pure-action profiles. We do not need to assume that proper subsets of players can use a joint randomization device.

Let $Z_m = \{0,1\}$ for each mobile player $m = 1,2,\ldots,M$ and $Z_i = \{0\}$ for each immobile player $i = M+1,\ldots,N$. This is the set of feasible "end/continue" votes for each player. Let also $Z := Z_1 \times \cdots \times Z_N$. For each $t = 0,1,\ldots$, let $H^t = [Z \times A]^t$ (where $A := A_1 \times \cdots \times A_N$ and $[Z \times A]^0$ is a singleton of the "null history") be the set of observed histories up to period t.

A pure strategy of a mobile player $m \in \mathcal{M}$ is a function (as long as the dynamic game continues)

$$s_m : H^t \to \{0,1\} \times A_m$$

and a pure strategy of an immobile player $i \in \{M+1,\ldots,N\}(= \mathcal{N} \setminus \mathcal{M})$ is

$$s_i : H^t \to A_i.$$

The equilibrium concept is subgame perfect equilibrium.

2.2 The extended minmax value

One might think that high payoff vectors sustained in the repeated games (by the "folk theorems", for example in Rubinstein, 1979, and Fudenberg and Maskin, 1986) are also sustainable even if the game must be endogenously repeated because high payoff vectors give incentives for players to continue the game forever. However, the equilibrium must include punishments, which impose a sufficiently low continuation payoff on each deviator. When players can leave an interaction at will, it is not guaranteed that we can enforce such a low continuation payoff. To be precise, for each feasible payoff vector, we need to find the **severest** punishment on each player, which may involve game termination, as well as to verify that the punishment is **incentive compatible** (Abreu, 1988).

For the stage game g^{st}, the minmax point is defined as usual (Fudenberg and Maskin, 1986). For each $j \in \mathcal{N}$, choose $\mu^{st,j} = (\mu_1^{st,j}, \ldots, \mu_N^{st,j}) \in A$ so that

$$(\mu_1^{st,j}, \ldots, \mu_{j-1}^{st,j}, \mu_{j+1}^{st,j}, \ldots, \mu_N^{st,j}) \in \arg \min_{a_{-j} \in A_{-j}} \left[\max_{a_j \in A_j} g_j^{st}(a_j, a_{-j}) \right],$$

and define

$$\underline{v}_j^{st} := \max_{a_j \in A_j} g_j^{st}(a_j, \mu_{-j}^{st,j}) = g_j^{st}(\mu^{st,j}).$$[12]

The one-shot actions $(\mu_1^{st,j}, \ldots, \mu_{j-1}^{st,j}, \mu_{j+1}^{st,j}, \ldots, \mu_N^{st,j})$ are called **stage-game minmax actions** by other players against player j. We call $\underline{v}_j^{st}$ player j's **stage-game minmax value**[13] and refer to $(\underline{v}_1^{st}, \ldots, \underline{v}_N^{st})$ as the **stage-game minmax point**.

To derive the appropriate minmax point for the repeated game with endogenous termination, we first extend the function g^{st}. Given a θ-majority rule, we define the extended payoff function for every player $n \in \mathcal{N}$, $g_n : Z \times A \to \mathbb{R}$, as follows.

$$g_n(\mathbf{z}, \mathbf{a}) = \begin{cases} g_n^{st}(\mathbf{a}) & \text{if } \sum_{j \in \mathcal{N}} z_j < \theta, \\ x_n & \text{if } \sum_{j \in \mathcal{N}} z_j \geqq \theta. \end{cases}$$

[12]The existence of $\mu^{st,j}$ (and $\underline{v}_j^{st}$), while it is not necessarily unique, is guaranteed by our assumptions of compact and convex action spaces and continuous payoff function g_j^{st}.

[13] If we interpret A_n as the set of mixed actions of player n, we need either (a) to assume that mixed actions are observable, or (b) to restrict attention to the observable minmax actions and corresponding stage-game minmax value. See Fudenberg and Tirole (1991), Chapter 5.1.

This g_n embeds the possibility that mobile players can choose to impose x_n on player n. Define $g : Z \times A \to \mathbb{R}^N$ by $g = (g_1, \ldots, g_N)$.[14]

Note that our repeated game with endogenous termination is not the repeated game with the stage game g. Once $\sum_{j \in \mathcal{N}} z_j \geqq \theta$ holds in some period, the game falls into the absorbing state, and all players receive x_n forever after, i.e., players cannot return to play g afterwards. Our model, in this way, captures the irreversible feature of game termination while the (infinitely) repeated g does not.[15]

For each $j \in \mathcal{N}$, choose $\mu^j = (\mu_1^j, \ldots, \mu_N^j) \in Z \times A$ such that

$$(\mu_1^j, \ldots, \mu_{j-1}^j, \mu_{j+1}^j, \ldots, \mu_N^j) \in \arg \min_{(\mathbf{z}_{-j}, \mathbf{a}_{-j}) \in [\times_{n \neq j} Z_n] \times [\times_{n \neq j} A_n]} \left[\max_{(z_j, a_j) \in Z_j \times A_j} g_j(z_j, \mathbf{z}_{-j}, a_j, \mathbf{a}_{-j}) \right],$$

and[16] (with a slight abuse of notation g_j)

$$\underline{v}_j = \max_{(z_j, a_j) \in Z_j \times A_j} g_j(z_j, a_j, \mu_{-j}^j) = g_j(\mu^j).$$

We call $\underline{v}_j$ player j's **extended minmax value** and refer to $(\underline{v}_1, \ldots, \underline{v}_n)$ as the **extended minmax point**. These are the relevant concepts for our folk theorems.

2.3 Characterization of the extended minmax value

We connect the stage-game minmax value $\underline{v}_n^{st}$ and the extended minmax value $\underline{v}_n$, which incorporates the possibility of endogenous termination. Below, we call a mobile player her and an immobile player him to distinguish them.

Lemma 1 *Assume that $N = M$, i.e., all players are mobile. For each $n \in \mathcal{N}$, her extended minmax value is*

$$\underline{v}_n = \begin{cases} x_n & \text{if } \theta = 1, \\ \min\{x_n, \underline{v}_n^{st}\} & \text{if } 1 < \theta < N, \\ \underline{v}_n^{st} & \text{if } \theta = N, \end{cases}$$

[14] g can be interpreted as a reduced form static game derived from the original two-stage game played in each period, as long as the game continues.

[15] Dutta (1995) derives the folk theorem for the general stochastic games with no absorbing state. His model, like the standard repeated games, lacks the irreversible feature and thus is different from ours.

[16] Since g_j^{st} is continuous and x_j is an exogenously fixed value, μ^j exists.

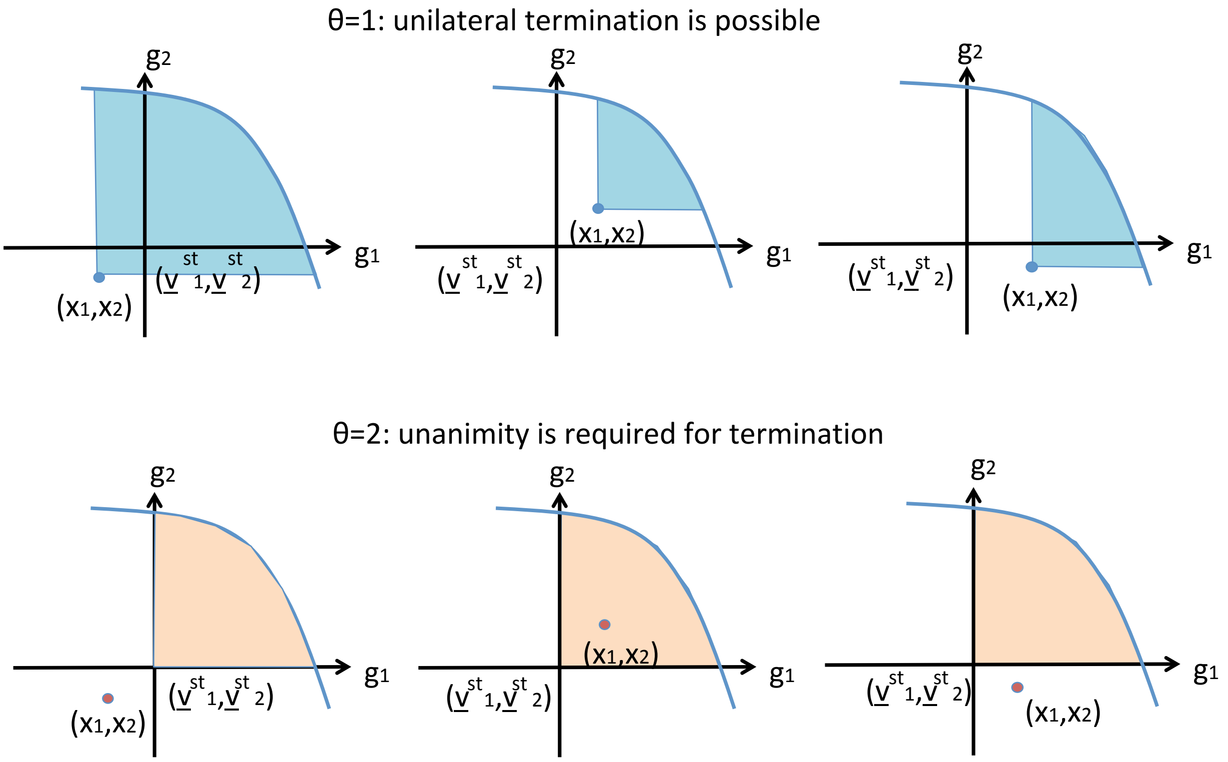

Figure 2: Individually rational regions when both players are mobile

where the middle case $1 < \theta < N$ *is vacuous when* $N = 2$.

Proof. If $\theta = 1$, then each (mobile) player can unilaterally terminate the game but cannot force the game to continue (since another player can choose to end the game). Hence each player can guarantee herself at least the payoff of x_n.

If $\theta = N$, then each player can force the game to continue by not choosing "end", but cannot unilaterally terminate the game. In this case a player can guarantee herself the in-game minmax payoff $\underline{v}_n^{st}$.

If $1 < \theta < N$, then no (mobile) player has a power to terminate or continue the game unilaterally. For each player n, $N - 1$ other players can impose x_n (by all of them choosing "end") and $\underline{v}_n^{st}$ (by continuing the game and minmaxing), whichever worse for n. ■

Figure 2 illustrates the regions of individually rational (IR) payoff vectors, which Pareto dominate the extended minmax point for two-player, all-mobile case, $N = M = 2$. When unilateral termination is possible ($\theta = 1$, top row), no player can be forced to receive less than her termination payoff. The IR set can be quite large if $(x_1, x_2) << (\underline{v}_1^{st}, \underline{v}_2^{st})$, expanding the relevant Pareto frontier as well (top left). This may happen when the outside situation of the

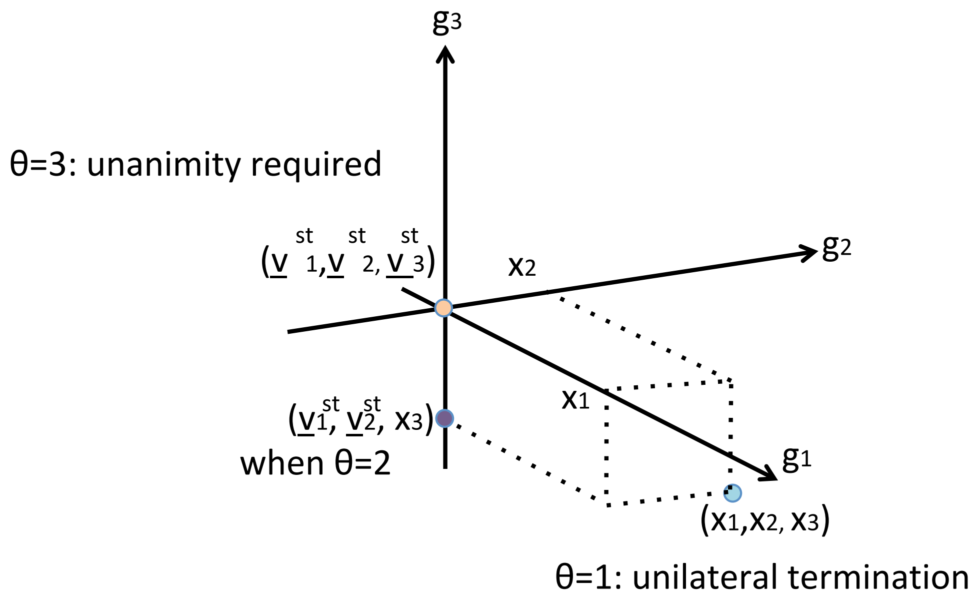

Figure 3: Extended minmax points for $\theta = 1, 2, 3$ when $x_3 < \underline{v}_3^{st}$ and $\underline{v}_n^{st} < x_n$ for $n = 1, 2$

community is very bad and a player can only secure a very low termination payoff. In this case, more unequal payoff vectors are individually rational as compared to the game without the termination option. In an asymmetric case (top right), the IR region expands for player 1 beyond the stage-game minmax level, because player 2 cannot guarantee her stage-game minmax level by termination. Hence, in a voluntary interaction, it is disadvantageous not to have a good outside option.

By contrast, when unanimity is required to terminate the game ($\theta = 2$, bottom row), no player can leave the interaction at will. Hence the IR region is the same as that of the ordinary repeated games, for any (x_1, x_2). This may be the reason that divorce requires a mutual agreement in many countries, to minimize the effect of termination payoffs.

When $M > 2$ (for $N \geqq 3$), there is an intermediate case such that $\theta \neq 1$ and $\theta \neq N$, which always (weakly) expands the IR region. Figure 3 illustrates extended minmax points for all possible θ when $N = M = 3$, $x_3 < \underline{v}_3^{st}$, and $\underline{v}_n^{st} < x_n$ for $n = 1, 2$. If $\theta = 1$, then each player can unilaterally terminate the game. Thus player $n = 1, 2$ can guarantee x_n, which is greater than her stage-game minmax payoff. The extended minmax point is the termination payoff combination (x_1, x_2, x_3), as in $N = 2$ case. If $\theta = N$, then each player can force the game to continue. Thus player 3 can avoid x_3 and the extended minmax point is the same as the stage-game minmax point $(\underline{v}_1^{st}, \underline{v}_2^{st}, \underline{v}_3^{st})$. The middle case of $\theta = 2$ allows players 1 and 2 to enforce x_3 on player 3. For players $n = 1, 2$, $\underline{v}_n^{st}$ can be imposed by other two players. Hence the extended minmax point is $(\underline{v}_1^{st}, \underline{v}_2^{st}, x_3)$, which is weakly Pareto-inferior to the extended minmax point of other θ's.

If there are immobile players in the community, the immobile players have the extended minmax level of $\min\{x_i, \underline{v}_i^{st}\}$ regardless of the parameters, and we have another possibility of the extended minmax value for mobile players, when the mobile player is unique.

Lemma 2 *Assume that* $N > M(\geqq 1)$.

For each immobile player $i \in \mathcal{N} \setminus \mathcal{M}$, *his extended minmax value is*

$$\underline{v}_i = \min\{x_i, \underline{v}_i^{st}\}, \ \forall\theta; \ 1 \leqq \theta \leqq M.$$

For each mobile player $m \in \mathcal{M}$, *if* $M \geqq 2$, *then her extended minmax value is*

$$\underline{v}_m = \begin{cases} x_n & \text{if } \theta = 1; \\ \min\{x_m, \underline{v}_m^{st}\} & \text{if } 1 < \theta < M; \\ \underline{v}_m^{st} & \text{if } \theta = M, \end{cases}$$

(where the middle case $1 < \theta < M$ *is vacuous when* $M = 2$.*)*

If $M = 1(= \theta)$, *then the unique mobile player's extended minmax value is*

$$\underline{v}_m = \max\{x_m, \underline{v}_m^{st}\}.$$

Proof. Each immobile player is in the same situation as a mobile player in Lemma 1 and when $1 < \theta < N(= M)$, since he cannot influence on the game termination nor continuation.

When $M \geqq 2$, each mobile player can unilaterally terminate the game (if $\theta = 1$) or unilaterally force the game to continue (if $\theta = M$) or has no such influence. Thus her extended minmax value has the same structure as in Lemma 1.

When $M = 1$, the unique mobile player m can do both unilateral termination and unilateral continuation of the repeated game. Hence she can guarantee $\max\{x_m, \underline{v}_m^{st}\}$ as her minmax value. ■

Lemmas 1 and 2 show that the severest punishment for each player, depending on the voting threshold, is fully characterized by the *static elements* of the game, and our extended minmax value are all explicitly calculated.[17]

An implication on mechanism design from Lemmas 1 and 2 is that, in order to have the maximal IR region, we should set $1 < \theta < M$.

2.4 Modified individual rationality

As usual, we define the set of **feasible** payoff vectors by[18]

$$\mathcal{F}^{st} = Conv\Big(\Big\{(v_1, \ldots, v_N) \mid \exists (a_1, \ldots, a_N) \in A_1 \times \cdots \times A_N \text{ with } g^{st}(a_1, \ldots, a_N) = (v_1, \ldots, v_N)\Big\}\Big).$$

For ordinary repeated games, the stage-game minmax point belongs to $\mathcal{F}^{st}$ by definition, and the "target" set of payoff vectors for the folk theorem (Fudenberg and Maskin, 1986) excludes those with some players receiving exactly the stage-game minmax value. In Figure 4, such set is the north east area of the red dashed lines.

By contrast, in the repeated game with endogenous termination, the extended minmax point can be outside of $\mathcal{F}^{st}$ since the termination payoffs are arbitrary. There is no hope to achieve the payoff vectors which belong to the "inefficient boundary" of $\mathcal{F}^{st}$, because we cannot construct a punishment equilibrium which must result in a Pareto-inferior payoff vector. Therefore, we focus on payoff vectors that have a Pareto-inferior and feasible payoff vector.

Henceforth we shall normalize the payoffs so that the extended minmax point is the origin, $(\underline{v}_1, \ldots, \underline{v}_N) = (0, \ldots, 0)$. Define the set of **modified individually rational payoff vectors** (assumed to be nonempty[19]) as follows.

$$V = \{(v_1, \ldots, v_N) \in \mathcal{F}^{st} \mid v_n > 0 \text{ for all } n \in \mathcal{N}; \\ \exists (v'_1, \ldots, v'_N) \in \mathcal{F}^{st};\ v_n > v'_n \text{ for all } n \in \mathcal{N}\}.$$

[17] This is due to our formulation of the termination payoff by the average payoff. In general, as in Dutta (1995), the minmax value depends on the overall structure of the dynamic game, and it is often difficult to solve the minmax value explicitly.

[18] The convex hull of a set X is denoted by $Conv(X)$. For an alternative definition of feasible payoff vectors that include the termination payoff vector, see Appendix B.

[19] By Lemmas 1 and 2, this assumption is satisfied for $\theta > 1$. When $\theta = 1$, V is empty if there is no feasible payoff vector which Pareto dominates $(x_1, \ldots, x_N)$. However, such stage games are not interesting because the unique equilibrium entails immediate termination.

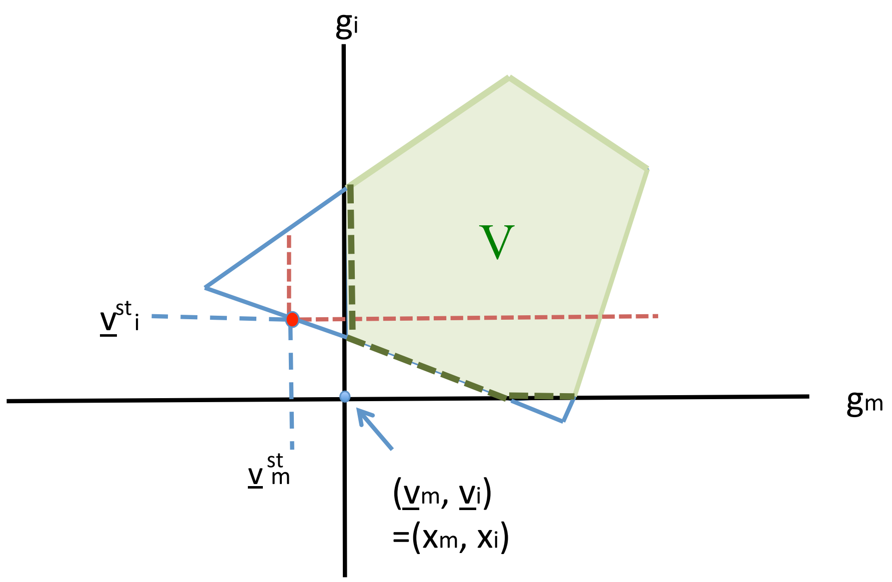

Figure 4: Modified IR payoff vectors for $N = 2$, $M = 1$

3 Immediately-ending and Never-ending Equilibria

The next lemma shows that, when $M \geqq 2$, unless the game has the unanimous ending rule ($\theta = M$), immediate termination is a subgame perfect equilibrium outcome after any history.

Lemma 3 *Assume $1 \leqq \theta \leqq M - 1$. For any period $t = 0, 1, 2, \ldots,$ and any history $h_t \in H^t$ (of the stage game), any (continuation) strategy combination in which at least $\theta + 1$ mobile players choose "end" at t constitutes a Nash equilibrium of the subgame starting at h_t.*

Proof. Suppose that at least $\theta + 1$ mobile players choose "end". This is possible by the assumption $\theta \leqq M - 1$. Then, a single mobile player cannot change the outcome by switching to "continue". Since no player can change her continuation payoffs, it is a (weakly) best response for each mobile player to choose "end" (and the game will be terminated immediately). ■

Therefore, a (continuation) strategy combination in which players choose a one-shot Nash equilibrium of g^{st} (as long as the game continues) and at least $\theta + 1$ mobile players choose "end" after any history is a subgame perfect equilibrium for any δ. We refer to an equilibrium in this class as an **immediately-ending equilibrium**.

An implication on mechanism design is that non-unanimous rules make termination an equilibrium and thus are useful to construct non-myopic equilibria. If there are only two mobile players, the unilateral-ending rule incentives players to terminate the game when it is needed.

The above argument is no longer true if $\theta = M$, since each mobile player can unilaterally change the outcome by switching her action from "end" to "continue", even if all other mobile players choose "end". Note that an immediately-ending equilibrium may be weakly dominated, since it holds regardless of $x_n \gtreqless \underline{v}_n^{st}$, but it is used only as a punishment, i.e., off the equilibrium path in the folk theorems. Note also that we do not need all of the mobile players to choose "end", when $\theta < M - 1$.

The next lemma shows that mobile players can also coordinate to continue the game if two or more mobile players need to agree on terminating the game, i.e., $\theta \geqq 2$.

Lemma 4 *Assume $2 \leqq \theta \leqq M$. Let σ^* be a subgame perfect equilibrium of the ordinary repeated game of g^{st} (without termination). Then, the strategy combination such that all mobile players choose "continue" in every period after any history and all players follow σ^* (as long as the game continues) constitutes a subgame perfect equilibrium of the repeated game with endogenous termination.*

Proof. Suppose that at most $\theta - 2$ mobile players choose "end" after any action history. (When $\theta = 2$, this means that all of mobile players choose "continue" always.) Then, a single mobile player cannot change the outcome by switching to "end". Since no player can change her continuation payoffs determined by σ^*, it is a (weakly) best response for each mobile player to choose "continue" and everyone to follow σ^*. ■

We call each equilibrium in this class a **never-ending equilibrium**. For each player n, the worst never-ending equilibrium for n is a candidate of the punishment against n's deviations. Applying the folk theorem of the ordinary repeated game of g^{st}, we can show that the payoff of the worst never-ending equilibrium can be made arbitrarily close to the stage-game minmax level. To see this, let V^{st} be all feasible payoff vectors that Pareto dominate the stage-game minmax point:

$$V^{st} = \{(v_1, \dots, v_N) \in \mathcal{F}^{st} \mid v_n > \underline{v}_n^{st} \text{ for all } n \in \mathcal{N}\}.$$

The following result is immediate from Lemma 4 and the folk theorem of Fudenberg and Maskin (1986).

Remark 1 *Assume $2 \leqq \theta \leqq M$ and that the dimensionality of V^{st} equals N. Then, for any $(v_1, \dots, v_N) \in V^{st}$, there exists $\underline{\delta} \in (0, 1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a* *subgame perfect equilibrium of the repeated game with endogenous termination in which player n's average payoff is v_n for all $n \in \mathcal{N}$.*

Hence the average payoff of the worst never-ending equilibrium for player n can be made arbitrarily close to $\underline{v}_n^{st}$ by making δ close to 1. Note, however, that whether the worst never-ending equilibrium exactly achieves[20] $\underline{v}_n^{st}$ or not depends on the stage game g^{st}. Under the assumption that V is nonempty, V^{st} has the full dimension (its dimensionality is the number of players, N) if and only if V has the full dimension.

4 Folk and Anti-Folk Theorems

4.1 Extended folk theorem for communities of mobile players

As the most standard case in economic applications, such as firm cartels and international alliances, suppose that all players are mobile ($M = N$). We obtain an extended folk theorem.

Theorem 1 *Assume that $M = N(\geqq 2)$ and the dimensionality of V equals N, the number of players.*[21] *Then, for any payoff vector $(v_1, \ldots, v_N) \in V$, there exists $\underline{\delta} \in (0, 1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player n's average payoff is v_n for all $n \in \mathcal{N}$.*

Proof. The proof is divided into two steps: when a non-unanimous ending rule such that $1 \leqq \theta \leqq N - 1$ is used and when termination requires unanimity, i.e., $\theta = N$.

Step 1: When $1 \leqq \theta \leqq N - 1$, the claim holds.

Proof of Step 1: Fix a target payoff vector $(v_1, \ldots, v_N) \in V$. By the definition of V, we can find a "target" action profile denoted by $\alpha = (\alpha_1, \ldots, \alpha_N) \in \Delta A$ so that $g(\alpha_1, \ldots, \alpha_N) = (v_1, \ldots, v_N)$ and there exists $(v'_1, \ldots, v'_N)$ in the interior of V such that $v_n > v'_n$ for all $n \in \mathcal{N}$. Since V has full dimension, there exists $\varepsilon > 0$ such that, for each j,

$$(v'_1 + \varepsilon, \ldots, v'_{j-1} + \varepsilon, v'_j, v'_{j+1} + \varepsilon, \ldots, v'_N + \varepsilon) \in V.$$

[20] A sufficient condition is that g^{st} has a Nash equilibrium which gives $\underline{v}_n^{st}$ to each player n (e.g., Prisoner's Dilemma).

[21] In other words, the interior of V relative to N-dimensional space is nonempty. For $N = 2$, we can drop this assumption as in Fudenberg and Maskin (1986), for both Theorem 1 and 2. See Appendix A.

Let $\beta^j = (\beta^j_1, \ldots, \beta^j_N) \in \Delta A$ be a "reward" action profile which attains this payoff vector (in expectation). Let $w^j_n = g^{st}_n(\mu^{st,j})$ be player n's per-period payoff when minmaxing player j and let $\hat{v}_n = \max_{a \in A} g^{st}_n(a)$ be player n's greatest one-shot payoff. For each n, choose an integer τ_n such that

$$\frac{\hat{v}_N}{v'_N} < 1 + \tau_n. \tag{1}$$

Now consider the following strategy for each player $m \in \mathcal{N}$ (we call a player m, so that we can reuse the same strategy in the proof of Theorem 2 for mobile players), starting at phase (A). Let $a^* \in A$ be a Nash equilibrium of the stage game g^{st}. The existence of a^* is guaranteed by our assumptions (i) and (ii) on g^{st}. Let us also write $z_{m,t} = 0$ as the "continue" action and $z_{m,t} = 1$ as the "end" action for any m and t.

(A) Choose "continue" and α_m each period as long as α was played last period. If a single player j deviates from (A),[22] then go to (B-j)-(end) or (B-j)-(minmax), depending on whether $x_j = 0$ (termination gives the extended minmax value) or not.

(B-j)-(end) If $x_j = 0 (= \underline{v}_j)$, choose "end". If the game continued somehow, play the one-shot Nash equilibrium action a^*_m and choose "end" in the next period after any observation.

(B-j)-(minmax) If $x_j \neq 0$, choose "continue" and $\mu^{st,j}_m$ for τ_j periods, and then go to (C-j).

(C-j) Choose "continue" and β^j_m there after.

If player j' deviates in the phase (B-j)-(minmax) or (C-j), then begin the appropriate phase (B-j'), depending on $x_{j'} = 0$ or not.

Let us show that no player deviates at the continue/end decision nodes. By the assumption of $\theta \leqq N - 1$ and Lemma 3, no (mobile) player deviates in (B-j)-(end) when everyone is supposed to choose "end". Consider the case when continuation is required, i.e., phases (A), (B-j)-(minmax), or (C-j), for some j. We have two possibilities.

Case 1: $x_n \neq 0 (= \underline{v}_n)$ holds for some player n.

In this case, Lemma 1 implies that $2 \leqq \theta$. Then, no single player can unilaterally change the continue/end outcome when all other players jointly choose "continue". Hence in this case no player deviates at the continue/end decision nodes.

Case 2: $x_n = 0$ holds for all n.

[22] If several players simultaneously deviate from (A), then we suppose that everyone ignores the deviation and continues to play α. The same remark applies to other phases than (A).

Notice that this does not happen in (B-j)-(minmax).[23] Also, if $2 \leqq \theta$, no player has an incentive to deviate to choose "end" as in Case 1. If $\theta = 1$ and in phases (A) or (C-j), if player n unilaterally ends the game, (s)he only receives payoff of 0. However, if (s)he follows the above strategy, (s)he receives a positive one-shot payoff (of action combination α or β^j) every period by the definition of V. Therefore, no player deviates to end.

It remains to check deviation incentives from the stage game actions. This part does not use the assumption that all players are mobile, and we do not need to check (B-j)-(end).

First, consider deviation incentives for any player n who has $x_n \neq 0 (= \underline{v}_n^{st})$, i.e., the extended minmax value is not the termination payoff but the stage-game minmax value. If such player n deviates from the action profile α in phase (A) and then conforms to the above strategy, (s)he receives at most $\hat{v}_n$ in that period, 0 for τ_n periods, and v'_n each period thereafter. Her total payoff is no greater than

$$\hat{v}_n + \frac{\delta^{\tau_n+1}}{1-\delta} v'_n.$$

If (s)he follows the above strategy, (s)he obtains $v_n/1-\delta$, so the net gain from a deviation is at most

$$\hat{v}_n + \frac{\delta^{\tau_n+1}}{1-\delta} v'_n - \frac{v_n}{1-\delta}.$$

By $v_n > v'_n$, this is less than

$$\hat{v}_n - \frac{1-\delta^{\tau_n+1}}{1-\delta} v'_n. \tag{2}$$

Since $\frac{1-\delta^{\tau_n+1}}{1-\delta}$ converges to $\tau_n + 1$ as δ tends to 1, condition (1) ensures that there exists $\underline{\delta}_n^A < 1$ such that (2) is negative for all $\delta > \underline{\delta}_n^A$.

If player n deviates in (B-n)-(minmax), i.e., when n is being punished, (s)he obtains at most 0 in that period, and then only lengthens her punishment, postponing the positive payoff v'_n. Hence n does not have a deviation incentive.

If player n deviates in (B-j)-(minmax), i.e., when another player $j(\neq n)$ is being punished, and then conforms, (s)he receives at most

$$\hat{v}_n + \frac{\delta^{\tau_n+1}}{1-\delta} v'_n,$$

[23] However, if there are immobile players in the community, it is possible that all mobile players have $x_m = 0$ but a deviator, who is an immobile player, has $x_j \neq 0$. See the proof of Theorem 2.

which is no greater than $\hat{v}_n + \frac{\delta}{1-\delta}v'_n$ for any $(1 \leqq)\ \tau \leqq \tau_n$. However, if (s)he does not deviate, (s)he receives

$$\frac{1-\delta^\tau}{1-\delta}w_n^j + \frac{\delta^{\tau+1}}{1-\delta}(v'_n + \varepsilon),$$

for some τ between 1 and τ_j, where τ is the number of remaining periods in (B-j)-(minmax). Therefore, the net gain from a one-step deviation is at most

$$\hat{v}_n + \frac{\delta}{1-\delta}v'_n - \{\frac{1-\delta^\tau}{1-\delta}w_n^j + \frac{\delta^{\tau+1}}{1-\delta}(v'_n + \varepsilon)\} = \hat{v}_n + \frac{1-\delta^\tau}{1-\delta}(\delta v'_n - w_n^j) - \frac{\delta^{\tau+1}}{1-\delta}\epsilon. \tag{3}$$

As $\delta \to 1$, the second term of the RHS in (3) remains finite because $\frac{1-\delta^\tau}{1-\delta}$ converges to τ, but the third term converges to negative infinity. Since there is a finite number of players, there exists $\underline{\delta}_n^B < 1$ such that for all $\delta > \underline{\delta}_n^B$, player n will not deviate in phase (B-j)-(minmax), for any $j \in \mathcal{N}$. Finally the argument for why players do not deviate in phase (C-j), for any j, is practically the same as that for phase (A). Therefore, there exists $\underline{\delta}_n < 1$ such that for all $\delta > \underline{\delta}_n$, player n will not deviate in phases (A), (B-j)-(minmax), and (C-j), for any $j \in \mathcal{N}$.

Second, consider deviation incentives for player n such that $x_n = 0$. If such player n deviates in the stage-game during any of the phases (A), (B-j)-(minmax), and (C-j) for any $j \in \mathcal{N}$, (s)he receives at most $\hat{v}_n$ in that period and the game will be immediately terminated. Hence the total payoff is no greater than $\hat{v}_n$. Instead, if (s)he conforms in each phase, (s)he receives $\frac{v_n}{1-\delta}$ in (A), $\frac{1-\delta^\tau}{1-\delta}w_n^j + \frac{\delta^{\tau+1}}{1-\delta}(v'_n + \epsilon)$ for some $1 \leqq \tau \leqq \tau_j$ in (B-j)-(minmax), and $\frac{v'_n+\varepsilon}{1-\delta}$ in (C-j). As $\delta \to 1$, each payoff when she conforms converges to positive infinity, but deviation payoff remains constant. Thus, there exists $\underline{\delta}_n < 1$ such that for all $\delta > \underline{\delta}_n$, player n will not deviate in any of phases (A), (B-j)-(minmax), and (C-j), for any $j \in \mathcal{N}$. This completes the proof of Step 1.

Step 2: When $\theta = N$, the claim holds.

Proof of Step 2: By Lemma 1, $\underline{v}_n^{st} = \underline{v}_n = 0$ for all $n \in \mathcal{N}(= \mathcal{M})$. Then, any payoff vector $(v_1, \ldots, v_N) \in V$ satisfies that $v_n > \underline{v}_n^{st}$. Modify the strategy combination in Step 1 in such a way that all players choose "continue" after any history. Since the stage-game action part of the above strategy combination is a subgame perfect equilibrium of the ordinary repeated game of g^{st} for sufficiently large δ by Fudenberg and Maskin (1986), Lemma 4 implies that the modified strategy combination is a subgame perfect equilibrium for sufficiently large δ. This completes the proof of Step 2 and the Theorem. ■

The key to the above extended folk theorem is that the severest punishment is always an equilibrium, when all players are mobile. If $\theta = 1$, each (mobile) player can guarantee herself the termination payoff, and hence the severest punishment on her is termination. Other mobile players can enforce this, because if at least one another mobile player chooses to end, no player can change the outcome and it is weakly optimal to choose end. If $1 < \theta \leqq M - 1$, then each (mobile) player n can guarantee herself to get $\min\{x_n, \underline{v}_n^{st}\}$ (Lemma 1). Therefore the severest punishment may be minmaxing (which requires a never-ending equilibrium), but that is enforceable because $1 < \theta$ means that other mobile players can force the game to continue (Lemma 4). Also if termination is the severest punishment, θ of other mobile players can jointly choose to end the game (Lemma 3). Finally, if $\theta = M$, termination is difficult to enforce (Lemma 3 does not cover this case). But the severest punishment on any (mobile) player is minmaxing (Lemma 1), and that is enforceable.

Theorem 1 showed that, when all players are mobile, termination is a part of an equilibrium, whenever it is needed to punish someone. When there are immobile players in the community, the case of $\theta = M$ requires a new strategy design to induce termination when it is needed to punish an immobile player but all mobile players have low termination payoffs. This problem is analyzed in details in Sections 4.3 and 5.

In terms of the range of equilibrium payoff vectors, the extended folk theorem achieves all feasible and modified individually rational payoff vectors. That may not imply that the set of equilibrium payoff vectors is smaller or larger than the one without the termination option (recall Figure 2), because the extended minmax point can be inside the stage-game feasible set $\mathcal{F}^{st}$. One clear implication is, however, that if the sustainable Pareto frontier expands, the newly attainable region must be in the direction of inequal payoff vectors, allowing a high payoff for some players at the expense of others receiving below the stage-game minmax value. This result gives a rationale to many economic phenomena in self-sustained communities. For example, in many immoral workplaces and organizations, threatening some members by firing may be used to coordinate on an unequal outcome. In a society where only men can divorce the wives but not vice versa and women have low outside options, the equilibrium outcome could be significantly favoring men. (moved from the Introduction.)

4.2 Extended folk theorem with non-unanimous ending rules

From now on we consider an arbitrary community with or without immobile players. Local communities and organizations can have players who cannot participate in the continue/end decision stage. For example, alien residents do not have the voting rights, and junior members of an organization are not allowed to sit in the "top" meetings to decide the fate of the organization. As we clarified after Theorem 1, whether termination is incentive compatible when it is needed to punish someone is the key to establish a folk theorem. Our second main result is that it is incentive compatible for any majority rule except the unanimous one ($\theta = M$).

Theorem 2 *Assume that the dimensionality of V equals N. For any $M \leqq N$, any θ such that $1 \leqq \theta \leqq M-1$ and any payoff vector $(v_1, \ldots, v_N) \in V$, there exists $\underline{\delta} \in (0,1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player n's average payoff is v_n for all $n \in \mathcal{N}$.*

Proof. It suffices to consider the case that $M < N$, i.e., there are immobile players. Fix a target payoff vector $(v_1, \ldots, v_N) \in V$. For mobile players, consider the same strategy as in Step 1 of the proof of Theorem 1. For each immobile player $i \in \mathcal{N} \setminus \mathcal{M}$, consider the following strategy, which extracts the stage-game action part of the mobile players' strategy.

(A) Play α_i each period as long as α was played last period. If player j deviates from (A), then go to (B-j).

(B-j)-(end) If $x_j = 0$, play a_i^* (as long as the game continues).

(B-j)-(minmax) If $x_j \neq 0$, play $\mu_i^{st,j}$ for τ_j periods, and then go to (C-j).

(C-j) Play β_i^j each period as long as the game continues.

If player j' deviates in phase (B-j) or (C-j), then begin phase (B-j').

The proof of stage-game incentives in Step 1 of Theorem 1 applies to any player.

For the continue/end decision nodes, the difference from the case of $M = N$ is that phase (B-j)-(minmax) needs to be checked and that we need to replace Lemma 1 with Lemma 2.

When ending is required, the assumption $1 \leqq \theta \leqq M-1$ and Lemma 3 imply that no mobile player deviates. When continuation is required and $x_m \neq 0 (= \underline{v}_m)$ holds for some mobile player m, by Lemma 2, $2 \leqq \theta$ ($\leqq M-1$) holds. Then, no single mobile player can unilaterally change the continue/end outcome and thus does not deviate. If $x_m = 0$ holds for all m and if $\theta = 1$, unilateral ending gives 0 to any mobile player m. Except the one-shot payoff w_m^j, all other one-shot payoffs in the relevant phases (payoffs from action combinations α and β^j) are positive by the definition of V. Hence there exists $\underline{\delta}_M \in (0,1)$ such that for all $\delta > \underline{\delta}_M$ and all $m \in \mathcal{M}$, the continuation payoff of player m is positive after any history. Therefore, no mobile player deviates to end. This completes the proof of Theorem. ■

4.3 Unanimous ending rule and anti-folk theorem

Recall that if all players are mobile (Theorem 1) or termination does not require unanimity (Theorem 2), then, whenever termination is needed to punish someone, it was a part of an equilibrium. By contrast, if there are immobile players and unanimity is needed to terminate the game, there can be a case in which termination is needed to punish someone but is not a part of any equilibrium. The next anti-folk theorem shows this case.

Theorem 3 *Assume $\theta = M$ and that there exist $m \in \mathcal{M}$ and $i \in \mathcal{N}\setminus\mathcal{M}$ such that $x_m < \underline{v}_m^{st}$ and $x_i = \underline{v}_i < \underline{v}_i^{st}$. Then, for any $\delta \in (0,1)$, there is no subgame perfect equilibrium of the repeated game with endogenous termination in which player i's average payoff v_i satisfies $v_i \in (\underline{v}_i, \underline{v}_i^{st})$.*

Proof. Suppose that there is a subgame perfect equilibrium in which player i's average payoff is $v_i \in (\underline{v}_i, \underline{v}_i^{st})$. Since player i can guarantee himself the minmax value of $\underline{v}_i^{st}$ when the game continues forever, the game must be terminated, at least with a positive probability, at some period. $\theta = M$ means that all mobile players must choose "end" simultaneously with a positive probability in that period. However, choosing "end" is not incentive compatible for the player m with $x_m < \underline{v}_m^{st}$, a contradiction. ■

Intuitively, the mobile player with $x_m < \underline{v}_m^{st}$ does not want to end the game, and the rule $\theta = M$ allows m to "block" termination of the game. Hence we cannot sustain a payoff less than the stage-game minmax value $\underline{v}_i^{st}$ of the immobile player(s). As Figure 5 illustrates, the anti-folk theorem is not necessarily bad for the society. It only excludes relatively low payoff vectors for the relevant immobile player i. However, that may reduce the maximal sustainable payoff level of another player.

[24]Whether $\underline{v}_m = x_m$ or not depends on the number of M, and thus we do not specify what $\underline{v}_m$ is.

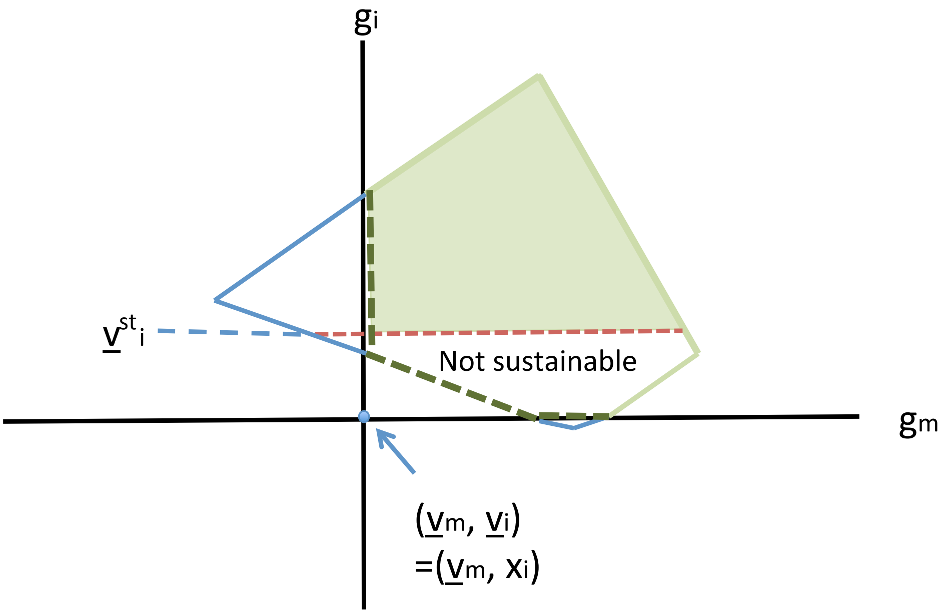

Figure 5: Anti-folk theorem[24]

4.4 When all immobile players have high termination payoffs

If we drop the assumption on immobile players in Theorem 3 and focus on the case that $x_i \geqq \underline{v}_i^{st}$, for all $i \in \mathcal{N} \setminus \mathcal{M}$, then termination is not needed to punish immobile players and we recover an extended folk theorem. The idea is as follows. If $\theta < M$, then Theorem 2 applies. If $\theta = M$ and $M \geqq 2$, then Step 2 of the proof of Theorem 1 shows that the severest punishment for each mobile player is an equilibrium, and min-maxing is an equilibrium to punish any immobile player. For the remaining case of $\theta = M = 1$, let $\mathcal{M} = \{1\}$. Lemma 2 implies that, for any payoff vector $(v_1, \ldots, v_N) \in V$, it holds that $v_1 > \underline{v}_1 = \max\{x_1, \underline{v}_1^{st}\}$. Hence min-maxing is enough to punish not only all immobile players but also the unique mobile player, and this can be done by the strategy combination constructed by Fudenberg and Maskin (1986).

Remark 2 *Assume that the dimensionality of V equals N, and that $x_i \geqq \underline{v}_i^{st}$, for all $i \in \mathcal{N} \setminus \mathcal{M}$. For any $M \leqq N$, any θ and any payoff vector $(v_1, \ldots, v_N) \in V$, there exists $\underline{\delta} \in (0, 1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player n's average payoff is v_n for all $n \in \mathcal{N}$.*

5 Termination incentives under the unanimous ending rule

Remark 2 showed that if we drop one of the assumptions of the anti-folk theorem 3 (namely, the existence of an immobile player who has $x_i < \underline{v}_i^{st}$), then the folk theorem is recovered. In this section, we consider dropping the other assumption of the anti-folk theorem 3 and assume that all mobile players have $x_m \geqq \underline{v}_m^{st}$ or $x_m > \underline{v}_m^{st}$. However, we allow immobile players to have arbitrary payoff structure. Hence the problem of making mobile players terminate the game to punish some immobile player is still present. Section 5.1 shows that we can use some never-ending equilibrium to motivate mobile players to terminate the game when it is needed. If there is a unique mobile player, however, the incentive mechanism in Section 5.1 does not work. For this case, Section 5.2 shows the scope of sustainable payoff vectors, by introducing a new concept of the extended effective minmax value.

5.1 Multiple mobile players

One way to make termination incentive compatible is to consider communities such that $M \geqq 2$ and the stage games such that the worst never-ending equilibrium can impose exactly the stage-game minmax value $\underline{v}_m^{st}$ on every mobile player simultaneously, such as the Prisoner's Dilemma (see footnote 20).

Theorem 4 *Assume $\theta = M \geqq 2$, that the dimensionality of V^{st} equals N, and that there exists $\hat{\delta} \in [0, 1)$ such that for any $\delta \in (\hat{\delta}, 1)$, the ordinary repeated game of g^{st} admits a subgame perfect equilibrium σ^* such that the equilibrium average payoff is $\underline{v}_m^{st}$ for all $m \in \mathcal{M}$. For any payoff vector $(v_1, \ldots, v_N) \in V$, there exists $\underline{\delta} \in (0, 1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player n's average payoff is v_n for all $n \in \mathcal{N}$, if $x_m \geqq \underline{v}_m^{st}$, $\forall m \in \mathcal{M}$.*

Proof. Note first that, by Lemma 2, the extended minmax value for each mobile player coincides with her stage-game minmax value.

Let the target action profile be α so that $(v_1, \ldots, v_N) = g(\alpha)$. Modify the strategy combination in the Proof of Step1 of Theorem 1 as follows. (For immobile players, omit the prescription of the continue/end decision nodes.)

(A) Choose "continue" and α_n each period as long as α was played last period. If a single player j deviates from (A), then go to (B-j).
(B-j)-(end) If $x_j = 0 (= \underline{v}_j)$, choose "end". If some mobile player m deviates in this phase to choose "continue", then play a never-ending equilibrium with the stage-game play path σ^* thereafter.

(B-j)-(minmax) If $x_j \neq 0$, choose "continue" and $\mu_n^{st,j}$ for τ_j periods, and then go to (C-j).

(C-j) Choose "continue" and β_n^j there after.

If player j' deviates in phase (B-j)-(minmax) or (C-j), then begin phase (B-j').

Lemma 4 assures that σ^* can indeed be a part of a never-ending equilibrium. Every mobile player receives $\underline{v}_m^{st}$ under σ^* by assumption. Since $x_m \geqq \underline{v}_m^{st}$ under the assumption, each mobile player m does not have a strict incentive to deviate from "end" to "continue" in (B-j)-(end).

When "continue" is prescribed, $\theta = M$ implies that no mobile player can change the outcome and thus does not deviate to end. Given this, no player deviates in the stage game when δ is sufficiently large. ■

The proof of Theorem 4 relies on the existence of a particular never-ending equilibrium σ^*. Note however that, even if such σ^* does not exist, Remark 1 guarantees that there exists a never-ending equilibrium which attains a (continuation) payoff vector arbitrary close to the stage-game minmax point when a discount factor is sufficiently large.

Now assume that $x_m > \underline{v}_m^{st}$ holds for all $m \in \mathcal{M}$, which is the strict inequality version of the assumption in Theorem 4. Then, Remark 1 implies that for any payoff vector $\tilde{v}$ such that $\tilde{v}_m \in (\underline{v}_m^{st}, x_m)$ for all $m \in \mathcal{M}$, there exists a never-ending equilibrium which attains this payoff vector $\tilde{v}$ when a discount factor is close to 1. Let $\tilde{\sigma}$ denote this equilibrium strategy combination. In the absence of σ^*, the players can use $\tilde{\sigma}$, instead of σ^*, to achieve an immediate game termination whenever they need to punish an immobile player i by terminating the game.

To sum up the above argument, we establish the following folk theorem.

Theorem 5 *Assume $\theta = M \geqq 2$ and that the dimensionality of V^{st} equals N. For any payoff vector $(v_1, \ldots, v_N) \in V$, there exists $\underline{\delta} \in (0,1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player n's average payoff is v_n for all $n \in \mathcal{N}$, if $x_m > \underline{v}_m^{st}$, $\forall m \in \mathcal{M}$.*

5.2 Unique mobile player and effective minmax payoff

The remaining case is that there is a unique mobile player, say player 1, and immobile players have an arbitrary payoff structure. Then the difficulty may arise when termination is needed to punish an immobile player. The proofs of Theorems 4 and 5 do not work because they explicitly rely on the assumption that there are multiple mobile players.

We construct a new class of subgame perfect equilibria, in which immobile players are divided into two groups: those who punish the mobile player if the game continues and those who do not. With this construction, the termination payoff of the mobile player need not be weakly greater than one-shot Nash equilibrium payoff.

Partition the immobile players as follows.

$$\mathcal{A} = \{i \in \mathcal{N} \setminus \mathcal{M} \mid x_i > \underline{v}_i^{st}\} \text{ and } \mathcal{B} = \{j \in \mathcal{N} \setminus \mathcal{M} \mid x_j \leqq \underline{v}_j^{st}\}.$$

All immobile players in $\mathcal{B}$ simultaneously receive their severest punishments if the game ends.[25] This means that if the game termination is expected to occur next period, any player $j \in \mathcal{B}$ has no incentive to choose an action that is not one-shot optimal.

Let $br : \times_{i \in \mathcal{A} \cup \{1\}} A_i \to \times_{j \in \mathcal{B}} A_j$ be the $\mathcal{B}$-group's **best reply mapping** such that for any $\mathbf{a}_{-\mathcal{B}} \in \times_{i \in \mathcal{A} \cup \{1\}} A_i$,

$$g_j^{st}(br(\mathbf{a}_{-\mathcal{B}}), \mathbf{a}_{-\mathcal{B}}) \geqq g_j^{st}(a_j, br(\mathbf{a}_{-\mathcal{B}})_{-j}, \mathbf{a}_{-\mathcal{B}}) \text{ for any } a_j \in A_j$$

simultaneously holds for all $j \in \mathcal{B}$.[26] That is, no immobile player in $\mathcal{B}$ can obtain an immediate deviation gain from the action profile $(br(\mathbf{a}_{-\mathcal{B}}), \mathbf{a}_{-\mathcal{B}})$. To put it differently, for each action profile of non-$\mathcal{B}$ players, $\mathbf{a}_{-\mathcal{B}}$, $br(\mathbf{a}_{-\mathcal{B}})$ essentially solves a Nash equilibrium of the modified game played only by $\mathcal{B}$ players, $g(\cdot, \mathbf{a}_{-\mathcal{B}})$. Since the existence of a Nash equilibrium is guaranteed in our environment, the mapping br must also be well-defined. Because there could be multiple best reply mappings, let

$$v_1^*(br) := \min_{\mathbf{a}_{\mathcal{A}} \in \times_{i \in \mathcal{A}} A_i} \max_{a_1 \in A_1} g_1^{st}(a_1, \mathbf{a}_{\mathcal{A}}, br(a_1, \mathbf{a}_{\mathcal{A}})), \tag{4}$$

and we define v_1^* as follows.

$$v_1^* := \min_{br} v_1^*(br).$$

By this definition, even if the group $\mathcal{A}$ is empty, v_1^* is the worst one-shot Nash equilibrium payoff on player 1. As $\mathcal{A}$ increases, v_1^* weakly decreases, and Remark 2 is the case when $\mathcal{A} = \mathcal{N} \setminus \mathcal{M}$.

[25] These immobile players are similar to the players who have "equivalent utilities" in the standard repeated games (Abreu, Dutta and Smith, 1994, and Wen, 1994).

[26] With a slight abuse of notation, we re-order the action combinations so that j's action is in the first coordinate.

Our new concept v_1^* extends the effective minmax payoff proposed by Wen (1994), and hence we call it **the extended effective minmax value**.

Our last theorem is that, if x_1 is greater than the one-shot severest punishment payoff v_1^* that other (immobile) players can impose on player 1, then the folk theorem obtains. Let $\mu^* = (\mu_1^*, \ldots, \mu_N^*)$ be the **extended effective minmax actions** against player 1, which achieves v_1^*. Note that $v_1^* \geqq \underline{v}_1^{st}$ is always satisfied.

Theorem 6 *Assume that $\mathcal{M} = \{1\}$ and that the dimensionality of V equals N. For any payoff vector $(v_1, \ldots, v_N) \in V$, there exists $\underline{\delta} \in (0,1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player n's average payoff is v_n for all $n \in \mathcal{N}$, if $x_1 \geqq v_1^*$ is satisfied.*

Proof. In view of Remark 2, we only need to consider the case that there exists an immobile player $i \neq 1$ such that $x_i < \underline{v}_i^{st}$.

Take any payoff vector $(v_1, \ldots, v_N) \in V$. Modify the strategy combination in the proof of Step 1 of Theorem 1 (for the mobile player) and the one in the proof of Theorem 2 (for immobile players) as follows. For any $j \in \mathcal{N}$, in the phase (B-j)-(end), the mobile player chooses "end". If the game continues in this phase, the players choose μ^* and the mobile player chooses "end" in the next period.

Note that, by definition $v_1^* \geqq \underline{v}_1^{st}$. Therefore, $x_1 \geqq v_1^*$ implies that x_1 ($\geqq v_1^*$) $\geqq \underline{v}_1^{st}$, and hence x_1 is the extended minmax value of player 1, i.e., $x_1 = \underline{v}_1 = 0$.

By construction, the mobile player 1 does not have an incentive to deviate from "end", if $x_1 \geqq v_1^*$, and hence immediate ending indeed constitutes a subgame perfect equilibrium. When continuation is required, the mobile player 1 does not deviate for sufficiently high δ's, because we can choose $v_1' > 0 = x_1$, when $(v_1, \ldots, v_N) \in V$. Also by construction, each $i \in \mathcal{B}$ has no incentive to deviate from μ^*.

For each $i \in \mathcal{A}$, if he deviates from μ^*, players go to (B-i)-(minmax) to minmax player i for τ_i periods, because $\underline{v}_i^{st}$ $(= 0) < x_i$ for this player. This completes the proof of Theorem 6. ■

Let us show how our v_1^* is connected to **the effective minmax payoff** proposed by Wen (1994). In his analysis, the minmax value for player n is modified from the standard minmax value if there exists another player j who has equivalent utilities to n.[27] Suppose that players n and j have equivalent utilities. Then, by definition, it is impossible to punish or reward n alone, separately from j, which implies that we cannot provide any incentive to player j to punish n. Consequently, the standard minmaxing definition must be modified accordingly. Let I_s be the set of players who have the equivalent utilities to n. Wen (1994) defines the effective minmax payoff of player n in game g^{st} (which we denote by v_n^W) as follows.

$$v_n^W := \min_a \max_{j \in I_s} \max_{a_j} g_n^{st}(a_j, a_{-j}). \tag{5}$$

Let a^W be the corresponding action profile that achieves v_n^W. Since all players in I_s have the equivalent utilities, they simultaneously take their best replies under a^W. Therefore, Wen's effective minmax payoff can be regarded as a special case of our extended effective minmax value. In fact, our definition (4) reduces to the definition (5) when we replace $\mathcal{A}$ with I_s and the unique mobile player with n.

To conclude, we showed four kinds of equilibria to induce termination whenever needed. One is to have multiple mobile players. The second one is to use a never-ending equilibrium which gives exactly the stage-game minmax level to each player (like the Prisoner's Dilemma). The third is to use a stage-game Nash equilibrium which gives a payoff not more than the termination payoff of the unique mobile player. The fourth is to divide immobile players and find the lowest equilibrium payoff that can be imposed on the unique mobile player by immobile players.

6 Sequential voting

So far, we considered simultaneous voting such that all mobile players make continue/end decisions without knowing other players' continue/end decisions. In this section, we alternatively consider a sequential voting process in which mobile players choose $z_m \in \{0, 1\}$ one by one before the (simultaneous-move) stage game. We assume that the dynamic game is immediately terminated at the moment when the total number of "end" reaches θ in the voting phase.

While this type of (strictly) sequential voting is a special case of dynamic voting models, we

[27]Players n and j have the equivalent utilities if one player's payoff is a positive affine transformation of the other's.

allow all possible sequence of voting order. That is, the following arguments do not depend on a specific order in which the mobile players vote.

We first define two group of mobile players: C-mobile players ($\mathcal{M}^C$) who strictly prefers game continuation to termination, and E-mobile players ($\mathcal{M}^E$) who possibly prefer ending the game to continuation. (Note that there may be other mobile players who have $x_m = \underline{v}_m^{st}$.)

$$\mathcal{M}^C = \{m \in \mathcal{M} \mid x_m < \underline{v}_m^{st}\} \text{ and } \mathcal{M}^E = \{m \in \mathcal{M} \mid x_m > \underline{v}_m^{st}\}$$

Each C-mobile player will never choose "end" if she alone can decide the game continuation, since choosing "continue" is strictly better than "end", no matter which equilibrium she would play in the continuing game. In our simultaneous voting model, even such a C-mobile player has a (weak) incentive to choose "end", if θ or more mobile players are expected to choose "end" and thus the game will be terminated irrespective of her individual vote. By contrast, in the sequential voting model, this type of coordinated vote can no longer be sustained in equilibrium when the number of C-mobile players is large.

Proposition 1 *For any sequence of voting order, if $M - |\mathcal{M}^C| < \theta$ holds, then there exists no subgame perfect equilibrium in which game termination occurs.*

Proof. If there exists an SPE in which game termination occurs, then there must be at least $|\mathcal{M}^C| - (M - \theta)$ of C-mobile players who choose "end", since $M - |\mathcal{M}^C| < \theta$. We show that this can never happen by induction.

Case 1 Suppose that $\theta = M$. Then termination occurs only if all C-mobile players choose "end". Note that the existence of C-mobile players is guaranteed by our assumption $M - |\mathcal{M}^C| < \theta$. Since each C-mobile player can deviate to "continue" to force the game continuation and improve her payoff from x_m to $\underline{v}_m^{st}$, termination never happens in equilibrium.

Case 2 Suppose $\theta = M - 1$, then termination occurs if and only if

(1) $|\mathcal{M}^C| - 1$ of C-mobile players choose "end" or

(2) all C-mobile players choose "end".

By our assumption $M - |\mathcal{M}^C| < \theta$, there exist at least two C-mobile players in this case. Consider the C-mobile player who votes in the earliest order among C-mobile players. If she chooses "continue", then termination occurs only if all the remaining C-mobile players choose "end". However, as we have shown in Case 1, termination never happens in this subgame since each C-mobile player has an incentive to choose "continue". Therefore, the existence of a game terminating SPE requires that the earliest C-mobile player must choose "end". However, the earliest C-mobile player can deviate to "continue" to force the game continuation, and hence termination never happens in this case.

Suppose that there exists no SPE in which game termination occurs from Case 1 through Case $k-1$. We shall show that termination never occurs in Case k, which concludes the proof.

Case k Suppose $\theta = M - (k-1)$, then termination occurs if and only if

(1) $|\mathcal{M}^C| - (k-1)$ of C-mobile players choose "end",

(2) $|\mathcal{M}^C| - (k-2)$ of C-mobile players choose "end",

$\vdots$

$(k-1)$ $|\mathcal{M}^C| - 1$ of C-mobile players choose "end" or

(k) All C-mobile players choose "end".

If the earliest C-mobile player chooses "continue", then termination occurs only if $|\mathcal{M}^C| - (k-2)$ of the remaining C-mobile players choose "end". However, termination never happens in this subgame because of our inductive assumption on Case $k-1$. Therefore, the existence of a game terminating SPE requires that the earliest C-mobile player must choose "end". However, the earliest C-mobile player has a deviation incentive to choose "continue", and hence termination never happens in case k as well. ■

Proposition 1 shows that termination may be impossible when a voting process is sequential, even when it was possible with simultaneous voting. However the key condition, $M - |\mathcal{M}^C| < \theta$, is difficult to be satisfied for small θ. For instance, if $\theta = 1$, the condition never holds unless all mobile players are C-mobile, i.e., $M = |\mathcal{M}^C|$.

When θ is small, players can credibly terminate the game even under sequential voting. To prove this, let us define $\overline{V}$, a subset of V such that each mobile player m receives more than the maximum of x_m and $\underline{v}_m^{st}$, and each immobile player i receives more than $\underline{v}_i^{st}$.

$$\overline{V} = \{(v_1, \ldots, v_N) \in V \mid v_e > x_e \text{ for all } e \in \mathcal{M}^E \text{ and } v_n > \underline{v}_n^{st} \text{ for all } n \in \mathcal{N} \setminus \mathcal{M}^E\}$$

Throughout this section, we assume that $\overline{V}$ is non-empty. Next proposition guarantees the existence of the immediate ending equilibrium.

Proposition 2 *Assume that the dimensionality of V equals N. For any sequence of voting order, if $|\mathcal{M}^E| \geqq \theta \geqq 2$, then there exists $\underline{\delta} \in (0,1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium in which the game ends in the initial period.*

The idea of the proof is as follows. Let all E-mobile players choose "end". If no one deviates, then the game ends before reaching the stage game, since we assume $|\mathcal{M}^E| \geqq \theta$. If some E-mobile player, say e, deviates, then let all remaining mobile players in that period choose "continue" and minmax e forever in the continuing game. Note that, by $\theta \geqq 2$, player e cannot escape from this punishment by unilaterally terminating the continuing game. Since her minmax payoff $\underline{v}_e^{st}$ is less than the termination payoff x_e by definition, e does not have an incentive to deviate from the beginning. The dimensionality condition of V guarantees that the punishment is credible, i.e., all the punishing players have an incentive to follow this punishing strategy. This mechanism has some similarly to the "expulsion room" system of some Japanese companies: if a firm wants a permanent employee to quit but (s)he does not agree to do so, (s)he will be assigned a meaningless task for a very long time.

Proof. Consider the following strategy combination starting in phase (A) for only mobile players.

(A) Every E-mobile player chooses "end" as long as no other E-mobile player has chosen "continue". Other mobile players choose "continue". If some E-mobile player deviates, then players move to phase (B1). Let j be the first E-mobile player who chooses "continue".[28]
(B1-j) All remaining mobile players choose "continue" in the end/continue decision node, and each player n follows $\mu_n^{st,j}$ in the stage game, and then go to (B2-j).
(B2-j) All mobile players choose "continue" in the end/continue decision nodes, and each player n follows $\mu_n^{st,j}$ in the stage games for $\tau_j - 1$ periods, and then go to (C-j).
(C-j) All mobile players choose "continue" in the end/continue decision nodes, and each player n follows β_n^j in the stage games there after.
If player j' deviates in the phase (B1-j), (B2-j) or (C-j), irrespective of whether the deviation occurs in an end/continue decision node or in the stage game, then begin the phase (B1-j').

[28]Note that if all E-mobile players follow (A), then the game ends when the θ-th E-mobile player chooses "end" and each player i receives termination payoff of x_i. If deviation is made by other mobile player than E-mobile, then players stay in phase (A).

Before considering the phases (B) and (C), let us define the following payoff set $\overline{V}_{-j}$.

$$\overline{V}_{-j} = \{(v_1, \dots, v_N) \in V \mid v_e > x_e \text{ for all } e \in \mathcal{M}^E \setminus \{j\} \text{ and } v_n > \underline{v}_n^{st} \text{ for all } n \in (\mathcal{N} \setminus \mathcal{M}^E) \cup \{j\}\}$$

This is a (weakly) larger set than $\overline{V}$ and two sets coincides when j is not E-mobile. Since $\overline{V}$ is non-empty by assumption, $\overline{V}_{-j}$ is also non-empty. Recall that $\mu_n^{st,j}$ is a n's action minimaxing player j in a stage game. Consider the following payoff vector $v^j := (v_1^j, \dots, v_N^j) \in \overline{V}_{-j}$ such that

$$v_j^j = \underline{v}_j^{st} + \varepsilon, v_e^j = x_e + 2\varepsilon \text{ for all } e \in \mathcal{M}^E \setminus \{j\} \text{ and } v_n^j = \underline{v}_n^{st} + 2\varepsilon \text{ for all } \mathcal{N} \setminus \mathcal{M}^E,$$

where ε is a small positive number and it satisfies $\underline{v}_j^{st} + \varepsilon < x_j$ if j is E-mobile. Note, by the full dimensionality of V (and thus $\overline{V}_{-j}$), there must exist such $\varepsilon > 0$ for each $j \in \mathcal{M}^E$. Let $\beta^j = (\beta_1^j, \dots, \beta_N^j) \in \Delta A$ be a "reward" action profile which attains this payoff vector v^j. Let $w_n^j = g_n^{st}(\mu^{st,j})$ be player n's per-period payoff when minmaxing player j and let $\hat{v}_n = \max_{a \in A} g_n^{st}(a)$ be player n's greatest one-shot payoff. For each n, choose an integer τ_n such that

$$\frac{\hat{v}_n}{\varepsilon} < 1 + \tau_n. \tag{6}$$

Let us first show that no E-mobile player has a deviation incentive in phase (A). If $j \in \mathcal{M}^E$ deviates in (A), then she receives $\underline{v}_j^{st}$ for τ_j periods in phase (B1-j) & (B2-j), and then receives $\underline{v}_j^{st} + \varepsilon$ ever since in phase (C-j). Her average payoff after deviation is close to $\underline{v}_j^{st}$ and is strictly less than her termination payoff x_j. Since she would receive x_j if she hadn't chosen deviation, following phase (A) is incentive compatible.

In phase (B1-j) & (B2-j), j takes best reply against minimaxing action profile taken by other players. So, if j deviates in a stage game, then she incurs deviation *loss*, and the punishment phase (B1-j) restarts. Clearly, she becomes worse off by choosing such deviation. Now suppose that some other player $n \neq j$ deviates in a stage game, then her total payoff becomes

$$\hat{v}_n + (\delta + \delta^2 + \dots + \delta^{\tau_n})\underline{v}_n^{st} + \frac{\delta^{\tau_n+1}}{1-\delta}(\underline{v}_n^{st} + \varepsilon) = \hat{v}_n + \frac{\delta}{1-\delta}\underline{v}_n^{st} + \frac{\delta^{\tau_n+1}}{1-\delta}\varepsilon, \tag{7}$$

which is strictly less than

$$\hat{v}_n + \frac{1}{1-\delta}(\underline{v}_n^{st} + \varepsilon).$$

If n is not E-mobile and follows the equilibrium punishment strategy, she obtains

$$(1 + \delta + \delta^2 + \cdots + \delta^{\tau-1})w_n^j + \frac{\delta^\tau}{1-\delta}(\underline{v}_n^{st} + 2\varepsilon) = \frac{1-\delta^\tau}{1-\delta}w_n^j + \frac{\delta^\tau}{1-\delta}(\underline{v}_n^{st} + 2\varepsilon), \tag{8}$$

for some τ between 1 and τ_j, where τ is the number of remaining periods in (B1-j) or in (B2-j). Therefore, the net gain from a one-step deviation is at most

$$\hat{v}_n + \frac{1}{1-\delta}(\underline{v}_n^{st} + \varepsilon) - \{\frac{1-\delta^\tau}{1-\delta}w_n^j + \frac{\delta^\tau}{1-\delta}(\underline{v}_n^{st} + 2\varepsilon)\} = \hat{v}_n + \frac{1-\delta^\tau}{1-\delta}(\underline{v}_n^{st} + \varepsilon - w_n^j) - \frac{\delta^\tau}{1-\delta}\varepsilon.$$

As $\delta \to 1$, the second term of the RHS remains finite because $\frac{1-\delta^\tau}{1-\delta}$ converges to τ, but the third term converges to negative infinity. So, there exists $\underline{\delta}_n^B < 1$ such that for all $\delta > \underline{\delta}_n^B$, player n will not deviate in phase (B1-j) or (B2-j), for any $j \in \mathcal{N}$. If n is E-mobile, then $\underline{v}_n^{st}$ is replaced by $x_n (> \underline{v}_n^{st})$ in (8). This implies that, other things being equal, E-mobile players have less deviation incentives than non-E-mobile players, and hence it suffices to check the incentive compatibility for non-E-mobile player.

Next, consider deviation incentives in phase (C-j). If $n \in \mathcal{N}$ deviates, her total payoff becomes (7). If she follows the equilibrium punishment strategy, she obtains $\frac{1}{1-\delta}v_n^j$. Note that, v_n^j is smallest if $n = j$ and hence it suffices to check the incentive compatibility for $n = j$. The net gain from a one-step deviation for j is

$$\hat{v}_j + \frac{\delta}{1-\delta}\underline{v}_j^{st} + \frac{\delta^{\tau_j+1}}{1-\delta}\varepsilon - \frac{1}{1-\delta}(\underline{v}_j^{st} + \varepsilon) = \hat{v}_j - \underline{v}_j^{st} - \frac{1-\delta^{\tau_j+1}}{1-\delta}\varepsilon.$$

As $\delta \to 1$, the last term of the RHS converges to $(\tau_j + 1)\varepsilon$, which is strictly more than $\hat{v}_j$ by (6). This implies that, for each $j \in \mathcal{N}$, there exists $\underline{\delta}_j^C < 1$ such that for all $\delta > \underline{\delta}_j^C$, player j will not deviate in phase (C-j).

Finally, check deviation incentives in end/continue decision in phases (B1-j), (B2-j) and (C-j). While player j may want to terminate the game in order to avoid the punishment, she needs some other mobile player(s) to choose "end". To verify this, first consider phases (B2-j) and (C-j). Since $\theta \geqq 2$, j cannot unilaterally terminate the game; at least $\theta - 1$ additional mobile players need to choose deviation in these situations. In (B1-j), it might be possible that a single mobile player, say m, can unilaterally terminate the game. This situation occurs only when $\theta - 1$ mobile players other than m have already chosen "end" before the game was into phase (B1-j). Note that m must be different from j, since j is the player who has just chosen "end", and hence she cannot make another continue/end decision in that period. Thus, we only need to check whether all mobile players other than j do not have incentives to choose "end".

It is clear that non-E-mobile players do not choose "end" since their termination payoffs are smaller than the minmax payoffs. For each E-mobile player $e \neq j$, as $\delta \to 1$ the average payoff when she follows the equilibrium punishment strategy converges to $x_e + 2\varepsilon$ which is strictly more than her termination payoff. So, player e does not have an incentive to terminate the game when δ is close to 1. This concludes the proof. ■

Based on this result, the folk theorem is straightforward. Since Proposition 2 establishes the existence of an immediately-ending equilibrium, the proof of Proposition 3 is essentially the same as that of Theorem 2 and hence we omit it.

Proposition 3 *Assume that the dimensionality of V equals N, $|\mathcal{M}^E| \geqq \theta \geqq 2$. For any sequence of voting order and any payoff vector $(v_1, \ldots, v_N) \in V$, there exists $\underline{\delta} \in (0,1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium in which player n's average payoff is v_n for all $n \in \mathcal{N}$.*

7 Concluding Remarks

Our theoretical results are as follows. We prove that, if all players are mobile, termination is an equilibrium whenever that is needed to punish someone. Hence an extended folk theorem holds for self-sustained community of mobile players, for any termination rule (Theorem 1). If mobile and immobile players co-exist, an extended folk theorem holds for any rule except the unanimous-ending rule (Theorem 2). When unanimity is required to end the game, an anti-folk theorem holds under some condition (Theorem 3). The sufficient condition is nearly necessary when there are more than one mobile player in the community. The most difficult case is when (i) there is a unique mobile player, (ii) termination is needed to punish someone, but (iii) the mobile player's termination payoff is less than her stage-game reservation value.

For this case, we extend the notion of the effective min-max payoff proposed by Wen (1994) to derive a modified folk theorem (Theorem 6). Finally, when voting is sequential, for any order of voting, an extended folk theorem holds if a punishment similar to Japanese "oidashi-beya" mechanism is feasible, i.e., the number of mobile players who prefer termination to continuation is not less than the minimum number of votes needed to terminate the game θ and $\theta \geqq 2$.

An interesting future extension is to allow x_n to depend on the history of the game. On one hand, $x_n(h_t)$ may become large as player n's accumulated payoff up to period t increases, that is, a good performance at the current position brings her a better outside offer. On the other hand, $x_n(h_t)$ may depend on how the game is terminated or the number of mobile players who choose "end". Consider a cartelized oligopoly market where the leniency program is implemented. If a single firm terminates the game (a cartel arrangement) by reporting to the regulatory authority, this firm gets a relatively high termination payoff through the amnesty rule. If multiple firms report simultaneously, each one gets a low termination payoff in expectation, since the fine reductions depend on which firm reports first, second and so on.[29]

However, the linkage between play paths and termination payoffs would make it difficult to derive the extended minmax value explicitly. Specifically, if the termination payoff is increasing in the accumulated payoff within the current game, the recursive structure is lost, which essentially changes our dynamic game. Nonetheless, if the extended minmax value for each player is well-defined, we expect that a similar analysis to this paper is possible.

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[29]The details of amnesty rules depend on countries and regions. The immunity of the fine accrues only to the first reporter in the U.S. In EU and Japan, the second and third firms also get reductions of fines.

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Appendix A: When There Are only Two Players

Our Theorem 2 imposes the dimensionality condition, but this can be dropped when there are only two players. The key insight, originally shown by Fudenberg and Maskin (1986), is that two players can take a simple punishment with "mutual minmax actions", which does not require the dimensionality condition. The next theorem establishes the corresponding result in our environment. The additional assumption $M = 2$ is imposed, since the folk theorem fails when $\theta = M = 1$ and the immobile player 2 has $x_2 < \underline{v}_2^{st}$, by Theorem 3.

Theorem 7 *Assume $N = M = 2$. Then, for any $(v_1, v_2) \in V$, there exists $\underline{\delta} \in (0, 1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player m's average payoff is v_m for $m = 1, 2$.*

Proof. By Lemma 1, (i) $\underline{v}_m = x_m$ if $\theta = 1$ $(< M)$, and (ii) $\underline{v}_m = \underline{v}_m^{st}$ if $\theta = 2$ $(= M)$.

Recall that player m's maximum possible payoff is denoted by $\hat{v}_m = \max_{a_1,a_2} g_m^{st}(a_1, a_2)$. Let $\alpha = (\alpha_1, \alpha_2) \in \Delta A$ be a target action profile so that $g(\alpha_1, \alpha_2) = (v_1, v_2) \in V$.

Case (i): $\theta = 1$ This is the case in which each mobile player can unilaterally terminate the game. Below we show that the following strategy starting at (A) for each $m \in \{1, 2\}$ attains $(v_1, v_2) \in V$, for large enough δ.

(A) Choose "continue" and then play α_m each period as long as (α_1, α_2) was played last period. After any unitary deviation from (A), go to (B).

(B) Choose "end".

By Lemma 3, immediate ending prescribed in (B) is incentive compatible, and it is the severest punishment for both players, since $\underline{v}_m = x_m$. Therefore, we only need to check deviation incentives from (A). If player m switches from "continue" to "end", she clearly becomes worse off since her lowest possible payoff (x_m) immediately realizes but no deviation gain arises by so doing. If she switches from α_m to any other action and then conforms, she receives at most $\hat{v}_m$ in that period, then 0 due to immediate ending; her total payoff is no greater than $\hat{v}_m$. If she conforms throughout, she obtains $v_m/1-\delta$, which converges to positive infinity as δ tends to 1. Thus, she does not have an deviation incentive for all δ larger than some $\underline{\delta} < 1$.

Case (ii): $\theta = 2$ This is the case in which no single mobile player can terminate the game. By Lemma 4, any feasible and individually rational payoff vector can be sustained, without the assumption on the dimensionality (by Fudenberg and Maskin, 1986), and the individual rationality coincides with the extended individual rationality.[30] ■

Appendix B: Alternative Set of Feasible Payoff Vectors

In the main text, we defined the feasible payoff vectors as those attainable by playing the stage game, $\mathcal{F}^{st} = Conv(g^{st}(A_1 \times \ldots \times A_N))$. This was because we wanted to investigate sustainability of the "ordinary repeated game outcomes" under the endogenous termination. An alternative definition of feasibility is to include the termination payoff vector, as follows.

$$\hat{\mathcal{F}} = Conv(\{(x_1, \ldots, x_N)\} \cup g^{st}(A_1 \times \ldots \times A_N)).$$

This is the set considered in the theoretical hypotheses of Wilson and Wu (2017)[31], and by public randomization before playing the game, any payoff vector in $\hat{\mathcal{F}}$ is feasible in the repeated game with endogenous termination.

However, since the repeated game with endogenous termination is *not equivalent to* the repeated game with the extended payoff function g, the modified individually rational payoff vectors in $\hat{\mathcal{F}}$ are not the correct "target" set for the folk theorem. This is because termination

[30]Under the conditions of Theorem 7, we do not need to exclude Pareto inefficient vectors from $\mathcal{F}^{st}$. In (i), any feasible payoff vector such that $v > 0$ strictly dominates the equilibrium payoff of immediate-ending, i.e., 0, and hence, any individually-rational but Pareto-inefficient payoff vectors in $\mathcal{F}^{st}$ can be sustained. In (ii), V is defined independently from x_m's, so the downward sloping part in Figure 4 does not exist.

[31]They restrict attention to Prisoner's Dilemma.

is the absorbing state, and if termination is not an equilibrium outcome, it is impossible to sustain a payoff vector which involves $(x_1, \ldots, x_N)$, i.e., the above "extended feasibly" does not expand the set of equilibrium payoff vectors.

Remark 3 *Assume that* $(x_1, \ldots, x_N) \notin \mathcal{F}^{st} = Conv(g^{st}(A_1 \times \ldots \times A_N))$. *For any payoff vector* $(v_1, \ldots, v_N) \in \hat{\mathcal{F}}$ *such that* $(v_1, \ldots, v_N) = \alpha(x_1, \ldots, x_N) + (1-\alpha)(w_1, \ldots, w_N)$ *for some* $\alpha \in (0,1)$ *and* $(w_1, \ldots, w_N) \in \mathcal{F}^{st}$ *but termination at* $t = 0$ *is not an equilibrium outcome, then* $(v_1, \ldots, v_N)$ *cannot be sustained by a subgame perfect equilibrium, for any* δ.

Proof. Suppose that $(v_1, \ldots, v_N)$ is sustained by a subgame perfect equilibrium σ. Then termination must occur on the equilibrium path of σ with a positive probability after some on-path history h. Since the termination payoffs $(x_1, \ldots, x_N)$ do not depend on the history, the players can reproduce the continuation strategy $\sigma\mid_h$ at $t = 0$ (after the null history). Therefore, if termination after a (public) history h is an equilibrium outcome, then termination in $t = 0$ is an equilibrium outcome, a contradiction. ■

This result holds for any public history model, if the termination payoffs are history independent. At least, we can show an extension of the folk theorem when termination at $t = 0$ is an equilibrium outcome. Let

$$\hat{V} := \Big\{(v_1, \ldots, v_N) \in Conv(\{(x_1, \ldots, x_N)\} \cup V), \mid v_n > 0 \text{ for all } n \in \mathcal{N}\Big\}.$$

Corollary 1 *Assume that the assumptions of Theorem 1 hold (thus immediate termination at* $t = 0$ *is an equilibrium outcome). For any payoff vector* $(v_1, \ldots, v_N) \in \hat{V}$, *there exists* $\underline{\delta} \in (0,1)$ *such that, for any discount factor* $\delta \in (\underline{\delta}, 1)$, *there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player n's average payoff is* v_n *for all* $n \in \mathcal{N}$.

Proof. Take an arbitrary $(v_1, \ldots, v_N) \in \hat{V}$. Then there exists a weight $\alpha \in [0,1]$ and a payoff vector $(w_1, \ldots, w_N) \in V$ such that $(v_1, \ldots, v_N) = \alpha(x_1, \ldots, x_N) + (1-\alpha)(w_1, \ldots, w_N)$. Since $(w_1, \ldots, w_N) \in V$ is sustainable for sufficiently large δ's by Theorem 1, consider a strategy profile such that at $t = 0$, players randomize immediate termination at $t = 0$ with probability α and the equilibrium strategy profile to sustain $(w_1, \ldots, w_N) \in V$ with probability $1 - \alpha$. Then this publicly randomized strategy profile is a subgame perfect equilibrium for the same δ's as those which make $(w_1, \ldots, w_N)$ an equilibrium payoff vector. ■